Active Calculus - Multivariable

Activities Workbook

Active Calculus - Multivariable

Activities Workbook

Nicholas Long
Stephen F. Austin State University

Mitchel T. Keller
University of Wisconsin-Madison

Contributing Authors

Steve Schlicker
Grand Valley State University

David Austin
Grand Valley State University

Matt Boelkins
Grand Valley State University

August 11, 2026

Edition: 2nd Edition

Website: `https://activecalculus.org/`

Contents

9 Precalculus of Multivariable Functions

9.2 Three Dimensional Space

Preview Activity 9.2.1. In this Preview Activity, we will recall some ideas about measurements and important ideas in two dimensional space.

(a) What are the coordinates of the point drawn in Figure 9.2.1? Draw segments from the point P to the vertical and horizontal axes to demonstrate the measurements of the coordinates.

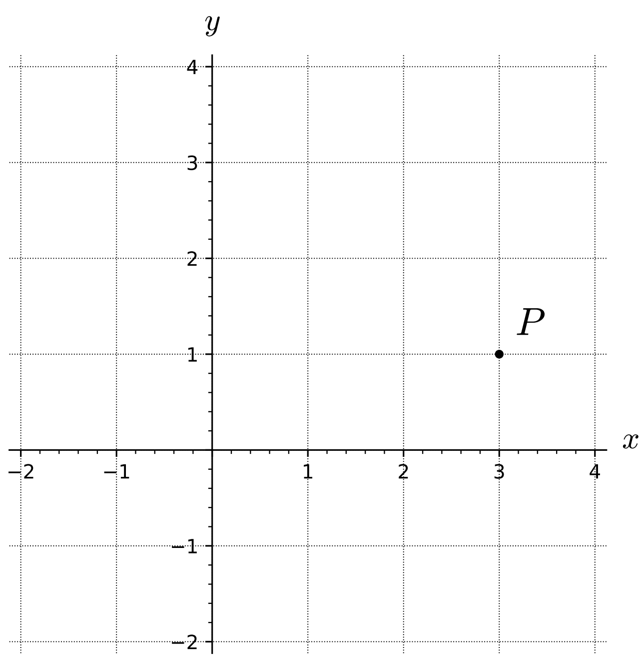


Figure 9.2.1 A 2D plot with point P labeled

(b) On the plot below, graph and label the following points: $P_1 = (0, 1)$, $P_2 = (1, 0)$, $P_3 = (2, -3)$, $P_4 = (3, -2)$, $P_5 = (-3, 2)$.

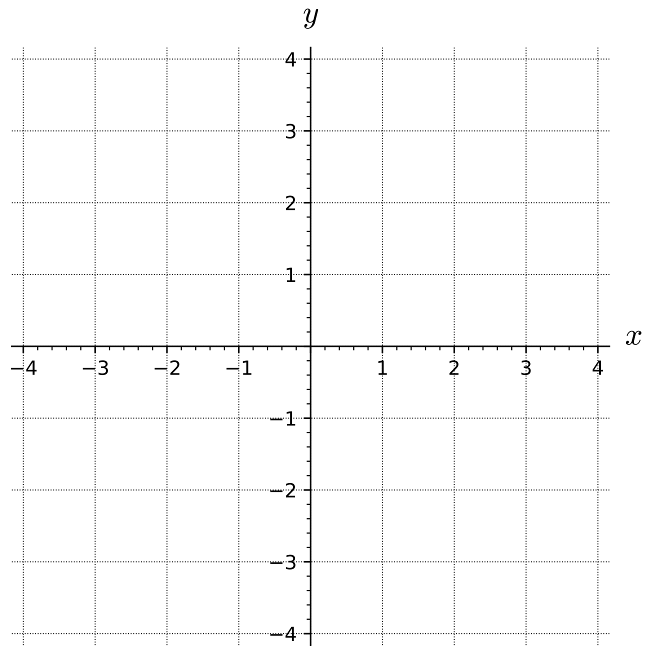


Figure 9.2.2 Two-dimensional axes for plotting points

(c) Give the coordinates of four points on the horizontal axis. What aspect do all of the points on the horizontal axis have in common? Use this idea to write an equation for the horizontal axis.

(d) On the axes below, draw a graph of each of the following equations: $x = 1$, $y = -2$, and $-x = y$

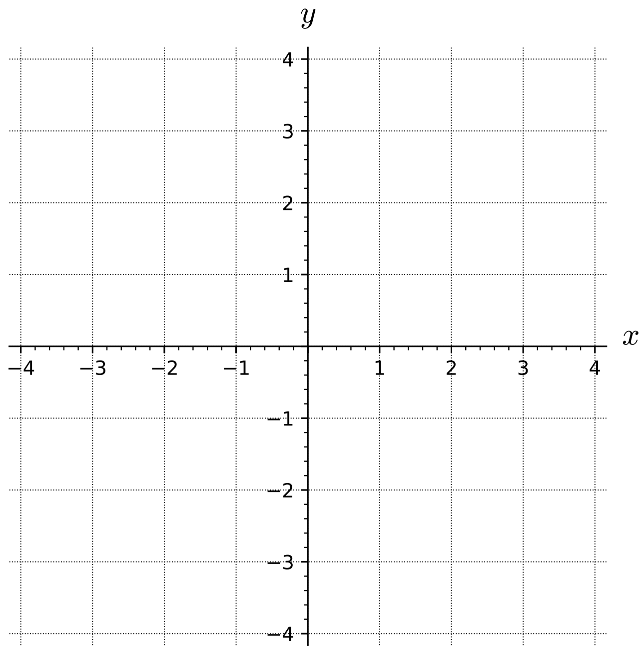


Figure 9.2.3 Two-dimensional axes for plotting equations

(e) Draw a plot of the points $(-1, 2)$ and $(7, -4)$. On your plot, draw the line segment that measures the distance between the given points and the segments that measure the horizontal and vertical changes. Your plotted segments should make a right triangle.

(f) Find the length of each of line segments in the plot that is your answer to part 9.2.1.e. Explain how the right triangle idea you drew can be generalized to find the distance between two points (x_1, y_1) and (x_2, y_2).

Activity 9.2.2 Drawing Points in 3D.. In this problem to move "forward" or "backwards" is to move in the direction of positive x (the x-coordinate increases, the y and z remain the same) or negative x (the x-coordinate decreases, the y and z remain the same), respectively; "to the right" or "to the left" is moving in the positive or negative y direction, respectively; "up" or "down" is moving in the positive or negative z direction, respectively:

(a) Find the coordinates of the point A where one ends if one starts at point $(1, 2, 3)$ and moves 5 units forward, 4 units to the left, and 2 units up.

(b) Draw the point A (that is, your answer to (a)) on a set of three-dimensional axes and include the line segments that show that coordinate's points as a set of directions from the origin (like in Figure 9.2.10 with the "Show Segments" option).

(c) Find the coordinates of the point B where one ends if one starts at point $(3, -4, 2)$ and moves 4 units backwards, 4 units to the right, and 4 units down.

(d) Draw the point B (that is, your answer to (c)) on a set of three-dimensional axes and include the line segments that show that coordinate's points as a set of directions from the origin (like in Figure 9.2.10 with the "Show Segments" option).

Activity 9.2.3.

(a) Consider the set of points (x, y, z) that satisfy the equation $x = 2$. Write a sentence to describe this set as best as you can.

(b) Consider the set of points (x, y, z) that satisfy the equation $y = -1$. Write a sentence to describe this set as best as you can.

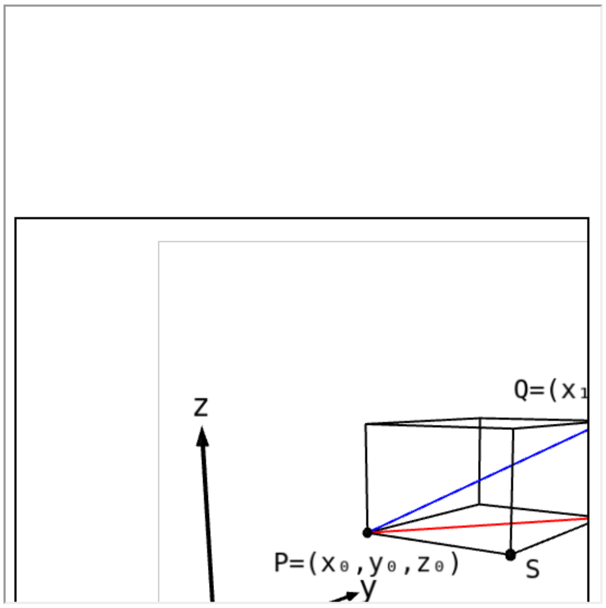
(c) Consider the set of points (x, y, z) that satisfy the equation $z = 0$. Write a sentence to describe this set as best as you can.

Activity 9.2.4. Remember that in two dimensions, the distance between $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ is

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

and is related to the pythagorean theorem applied to a right triangle that measures changes in each coordinate direction (as demonstrated in part 9.2.1.f).

Let $P = (x_0, y_0, z_0)$ and $Q = (x_1, y_1, z_1)$ be two points in $\mathbb{R}^3$. These two points form opposite vertices of a rectangular box whose sides are fundamental planes as illustrated in Figure 9.2.14, and the distance between P and Q is the length of the blue diagonal shown in Figure 9.2.14.



Standalone

In Context

Embed

Figure 9.2.14 A plot of $P = (x_0, y_0, z_0)$ and $Q = (x_1, y_1, z_1)$ with connecting segments

(a) Consider the right triangle PRS in the base of the box whose hypotenuse is shown as the red line in Figure 9.2.14. What are the coordinates of R and S?

(b) Give the equation of the fundamental plane that contains the right triangle PRS.

(c) Since the right triangle PRS lies in a plane, we can use the Pythagorean Theorem to find a formula for the length of the hypotenuse of this triangle. Find the length of the segment PR in terms of x_0, y_0, x_1, and y_1.

(d) Triangle PRQ has hypotenuse drawn with as blue segment connecting the points P and Q. Segment PR, which is the hypotenuse of triangle PRS that we considered earlier, is a leg of triangle PRQ. This triangle lies entirely in a plane, so we can again use the Pythagorean Theorem to find the length of its hypotenuse. Show that the length of PQ is

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}.$$

9.3 Vectors

Preview Activity 9.3.1. Postscript is a programming language whose primary purpose is to specify how to generate text or graphics. The following is a simple set of Postscript commands that produces the triangle in the plane with vertices $(0, 0)$, $(1, 1)$, and $(1, -1)$:

```
(0,0) moveto
(1,1) lineto stroke
(1,-1) lineto stroke
(0,0) lineto stroke
```

The process described by these commands is

1) tell Postscript to start at the point $(0, 0)$,

2) draw a line from the point $(0, 0)$ to the point $(1, 1)$ (this is what the `line to` and `stroke` commands do),

3) draw lines from $(1, 1)$ to $(1, -1)$, and

4) draw from $(1, -1)$ back to the origin.

Each of these commands encodes two important pieces of information: a direction in which to move and a distance to move. Mathematically, we can capture this information succinctly in a vector. To do so, we record the movement on the map in a pair $\langle x, y\rangle$, where x is the horizontal displacement and y the vertical displacement from one point to another. This pair $\langle x, y\rangle$ is a **vector**, and we call each number in the pair a **component**. As an example, the vector from the origin to the point $(1, 1)$ is represented by $\langle 1, 1\rangle$ while the vector from the point $(1, 1)$ to the origin is represented by $\langle -1, -1\rangle$.

(a) What is the vector $\vec{v}_1 = \langle x, y\rangle$ that describes the displacement from the point $(1, 1)$ to the point $(1, -1)$?

(b) How can we use the two components of $\vec{v}_1$ to determine the distance from the point $(1, 1)$ to the point $(1, -1)$?

(c) Suppose we want to draw the triangle with vertices $A = (2, 3)$, $B = (-3, 1)$, and $C = (4, -2)$. As a shorthand notation, we will denote the vector from the point A to the point B as $\overrightarrow{AB}$. Determine the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{AC}$.

(d) How are the horizontal displacements of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ related to the horizontal displacement of $\overrightarrow{AC}$? Does the same relationship follow for the vertical displacements? Write a couple of sentences to explain your reasoning in the context of this problem.

Activity 9.3.2. After being bored in your previous calculus class (and definitely not in multivariable calculus) you count the floor tiles to see how large the room is. You now know that the room is 32 feet wide and 42 feet in length. Additionally, your tall friend Chuck can barely jump and touch the ceiling which means the ceiling is 10 feet high. Your calculus professor notices you doing these measurements and decides to create a classroom coordinate system. Your professor walks to the center of the room and notices that their head is five feet above the ground and sets their head as the origin of the classroom coordinate system. Your friend Alice is sitting at $A = (9, -6, -2.5)$, a projector is located at position $B = (0, 1, 3)$, and your friend Carlos is working while sitting on the floor at $C = (-2, 20, -5)$. All distances are measured in feet.
For each of the directed line segments given below, determine the components of the indicated vectors and explain in context what each represents.

(a) $\overrightarrow{OA} = \langle\ ,\ ,\ \rangle$

(b) $\overrightarrow{OB} = \langle\ ,\ ,\ \rangle$

(c) $\overrightarrow{OC} = \langle\ ,\ ,\ \rangle$

(d) $\overrightarrow{AB} = \langle\ ,\ ,\ \rangle$

(e) $\overrightarrow{AC} = \langle \, , \, , \, \rangle$

(f) $\overrightarrow{BC} = \langle \, , \, , \, \rangle$

Activity 9.3.3. In this problem, we will be navigating to help find a friend who is lost at a local state park. We will be navigating using traditional map coordinates with east going in the positive x-direction and north going in the positive y-direction.

(a) Your friend told you they would be staying 3 km east and 4 km north of the main parking lot. You drive to the parking lot and park next to your friend's car, then hike 3 km east and 4 km north. You get to the location you expected your friend to be at, but you don't find your friend and call the ranger station. The ranger station says they think your friend is 2 km west and 1 km north of your current location. We will denote the location of the car with point C, your friend's originally anticipated location as point A, and the ranger's suggested location as point S.

On the grid below, we have labeled point C. Draw the location of each of the other points A and S.

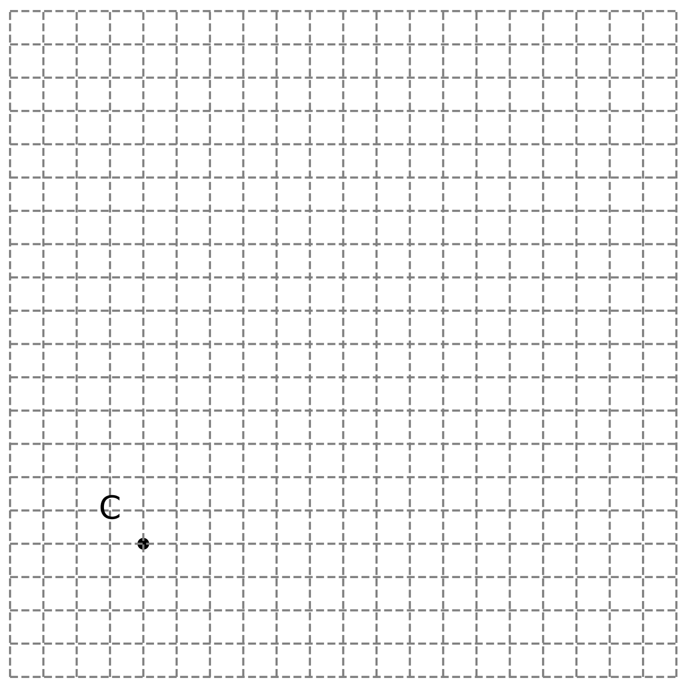


Figure 9.3.5 A grid for mapping locations needed to find your friend

(b) Using your picture from the previous task, give the components of the following vectors:

(a) $\overrightarrow{CA}$

(b) $\overrightarrow{CS}$

(c) $\overrightarrow{AS}$

(c) In the context of this problem, explain why you can add the horizontal components of $\overrightarrow{CA}$ and $\overrightarrow{AS}$ to get the horizontal component of $\overrightarrow{CS}$. Write a sentence or two about why this argument should work for the vertical components as well.

(d) After hiking to the location suggested by the ranger, you still don't see your friend and call the ranger station again. The regional manager of rangers answers this time. She says the first ranger made a mistake in their navigation. They sent a drone to the location of your car and the drone spotted your friend along the same direction from your car to point A, but your friend is three times as far away from your car as you were when you were at point A.

In order to avoid confusion or any other mistakes, you want to compute the vector from your car to the drone's suggested location, which we will call D. What are the components of $\overrightarrow{CD}$? Write a sentence or two to compare this vector to $\overrightarrow{CA}$.

(e) So we need to get from point S to point D. Explain how to use the components of $\overrightarrow{CS}$ and $\overrightarrow{CD}$ to find the components of $\overrightarrow{SD}$.

(f) You find your friend at point D and want to take them back to your car to go home. Explain how you can use the components of $\overrightarrow{CD}$ to give directions from point D back to your car.

Activity 9.3.4. In this activity, use Properties of vector operations to simplify each of the following expressions into a single vector of the form $\langle a, b, c\rangle$. For this activity, let $\vec{u} = \langle 1, -2, 3\rangle$, $\vec{v} = \langle 2, -2, 5\rangle$, and $\vec{w} = \langle -3, 0, 3\rangle$.

(a) $\vec{u} + 2\vec{v} + 3\vec{w}$

(b) $2\vec{w} - 3\vec{u}$

(c) $-4(\vec{v} - \vec{w})$

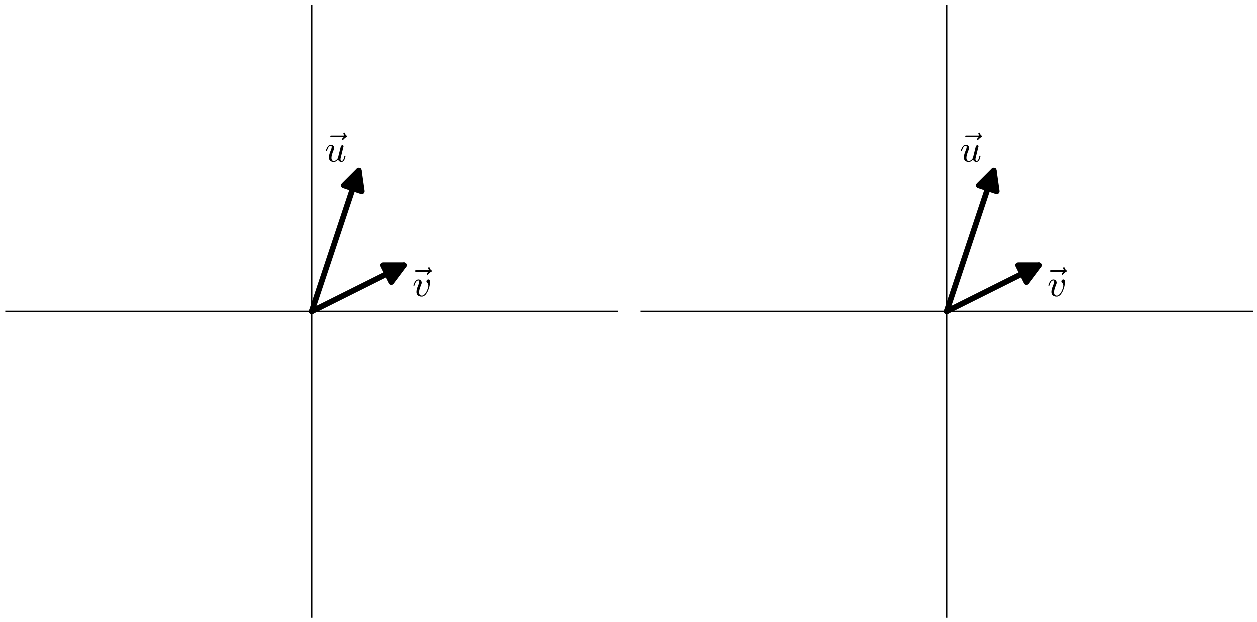
(d) $-2(3\vec{u})$

(e) Give the components of the vector that fills in the blank: $(2\vec{w} - 3\vec{u}) - \underline{\qquad\qquad\qquad} = \vec{0}$

(f) Explain why the following expression does not make sense:

$$2\langle 2, 4, -7\rangle - 3\langle -\pi, \sqrt{7}\rangle$$

Activity 9.3.5. Suppose that $\vec{u}$ and $\vec{v}$ are the vectors shown in Figure 9.3.9.



(a) Sketch sums **(b)** Sketch multiples

Figure 9.3.9

(a) On the axes in Figure 9.3.9(a), sketch the vectors $\vec{u}+\vec{v}$, $\vec{v}-\vec{u}$, $2\vec{u}$, $-2\vec{u}$, and $-3\vec{v}$.

(b) What are the components of $0\vec{v}$?

(c) On the axes in Figure 9.3.9(b), sketch the vectors $-3\vec{v}$, $-2\vec{v}$, $-1\vec{v}$, $2\vec{v}$, and $3\vec{v}$.

(d) Give a geometric description of the set of terminal points of the vectors $t\vec{v}$ where t is any scalar.

(e) On the set of axes in Figure 9.3.9(b), sketch the vectors $\vec{u} - 3\vec{v}$, $\vec{u} - 2\vec{v}$, $\vec{u} - \vec{v}$, $\vec{u} + \vec{v}$, and $\vec{u} + 2\vec{v}$.

(f) Give a geometric description of the set of terminal points of the vectors $\vec{u} + t\vec{v}$ where t is any scalar.

Activity 9.3.6.

(a) Let $\vec{u} = \langle 2, 3\rangle$ and $\vec{v} = \langle -1, 2\rangle$. Find $\|\vec{u}\|$, $\|\vec{v}\|$, and $\|\vec{u}+\vec{v}\|$. Is it true that $\|\vec{u}+\vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$?

(b) Under what conditions will $\|\vec{w}_1 + \vec{w}_2\| = \|\vec{w}_1\| + \|\vec{w}_2\|$?

(c) With the vector $\vec{u} = \langle 2, 3\rangle$, find the lengths of $2\vec{u}$, $3\vec{u}$, and $-2\vec{u}$, respectively, and use proper notation to label your results.

(d) In general, if t is any scalar, how will $\|t\vec{w}\|$ be related to $\|\vec{w}\|$?

(e) Of the vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{\imath} + \hat{\jmath}$, which are unit vectors?

(f) Find a unit vector $\vec{v}$ whose direction is the same as $\vec{u} = \langle -2, 3 \rangle$.

(g) Find a unit vector $\vec{v}$ in the opposite direction to $\vec{u} = \langle -2, 3 \rangle$.

9.4 The Dot Product

Preview Activity 9.4.1. In this activity, we will use the following vectors:

$$\vec{u}_1 = \langle 1, 2\rangle, \vec{u}_2 = \langle -1, 1\rangle, \vec{u}_3 = \langle 2, -1\rangle, \vec{u}_4 = \langle 1, -2\rangle$$

(a) Draw representatives of each of $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$ in standard position using the axes below.

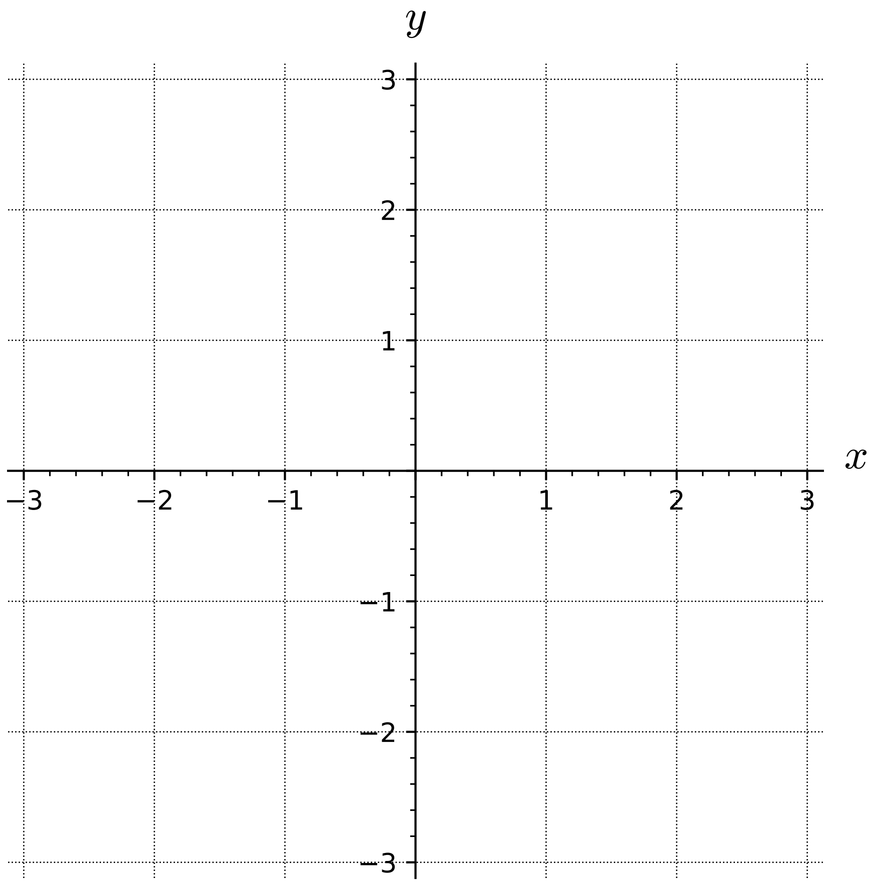


Figure 9.4.1 A plot of the 2D plane

(b) Compute each of the dot products listed in the table below. Fill in only the "Value" column at this point. The "Angle Classification" column will be completed in the next part.

Dot Product	**Value**	**Angle Classification**
$\vec{u}_1 \cdot \vec{u}_2$		
$\vec{u}_1 \cdot \vec{u}_3$		
$\vec{u}_1 \cdot \vec{u}_4$		
$\vec{u}_2 \cdot \vec{u}_4$		
$\vec{u}_3 \cdot \vec{u}_4$.		

(c) When we look at vectors drawn with the same initial point, as you have done with the vectors in Figure 9.4.1, we can consider the angle between two vectors to be the *smaller* angle formed by looking at them. This results in an angle between 0 and π if we measure in radians or between $0°$ and $180°$ if we measure in degrees. Rather than measuring the angles between the vectors you've drawn in Figure 9.4.1 precisely, classify the angle between the pairs of vectors in the table in part (b) by writing acute, right, or obtuse in the "Angle Classification" column.

(d) Look at the values of the dot products that you computed in part (b) and the classifications of the angles between the vectors that you made in part (c). Write a sentence or two to describe what you notice about the relationship between the *sign* of the dot product and the type of angle between the vectors.

(e) Does the value of the dot product or the classification of the angle change if we change the order in which we consider the vectors?

(f) Do you think that knowing only the value of the dot product $\vec{v} \cdot \vec{w}$ would be enough to determine the exact value of the angle between $\vec{v}$ and $\vec{w}$ if you didn't know the components of $\vec{v}$ and $\vec{w}$? Write a sentence to explain your reasoning.

(g) Although the angle between a vector and itself is 0, the dot product of a vector with itself can be computed. Compute $\vec{u}_1 \cdot \vec{u}_1$ and $\vec{u}_2 \cdot \vec{u}_2$ as well as the magnitude of $\vec{u}_1$ and $\vec{u}_2$. What do you notice about the relationship between these values? Write a sentence explaining why this seems reasonable based on the way the dot product of a vector with itself and the magnitude of that vector are computed.

Activity 9.4.2.

(a) Determine each of the following.

(i) $\langle 1, 2, -3\rangle \cdot \langle 4, -2, 0\rangle$

(ii) $\langle 0, 3, -2, 1\rangle \cdot \langle 5, -6, 0, 4\rangle$

(b) Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$. Suppose that you know that $\vec{u} \cdot \vec{w} = 10$ and $\vec{v} \cdot \vec{w} = -3$. Compute $(\vec{u} + \vec{v}) \cdot \vec{w}$.

(c) Let $\vec{u}, \vec{v}$ be vectors in $\mathbb{R}^n$. Suppose that you know that $\vec{u} \cdot \vec{v} = 3$, $\|\vec{u}\| = 4$, and $\|\vec{v}\| = 7$. Compute $(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$.

(d) Let $\vec{u}, \vec{v}$ be vectors in $\mathbb{R}^n$. For what value of t is $(t\vec{u} + \vec{v}) \cdot \vec{v} = 0$?

Activity 9.4.3. Determine each of the following:

(a) The length of the vector $\vec{u} = \langle 1, 2, -3 \rangle$ using the dot product.

(b) The angle between the vectors $\vec{u} = \langle 1, 2 \rangle$ and $\vec{v} = \langle 4, -1 \rangle$ to the nearest tenth of a degree.

(c) The angle between the vectors $\vec{y} = \langle 1, 2, -3 \rangle$ and $\vec{z} = \langle -2, 1, 1 \rangle$ to the nearest tenth of a degree.

(d) If the angle between the vectors $\vec{u}$ and $\vec{v}$ is a right angle, what does the expression $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos(\theta)$ say about their dot product?

(e) If the angle between the vectors $\vec{u}$ and $\vec{v}$ is acute—that is, less than $\pi/2$—what does the expression $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos(\theta)$ say about their dot product?

(f) If the angle between the vectors $\vec{u}$ and $\vec{v}$ is obtuse—that is, greater than $\pi/2$—what does the expression $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos(\theta)$ say about their dot product?

Activity 9.4.4. Return to Preview Activity 9.4.1 and use the results of this subsection to determine which pairs of vectors are orthogonal. Also classify the angles between non-orthogonal pairs of vectors as acute or obtuse.

Activity 9.4.5. In Figure 9.4.8, you can see illustrations of splitting $\vec{u}$ into two parts: $\vec{w}_1$, which is parallel to $\vec{v}$, and $\vec{w}_2$, which is orthogonal to $\vec{v}$. Use this figure for reference as you do the following.

(a) We know from the previous subsection that there is a third configuration of vectors, which occurs when $\vec{u}$ and $\vec{v}$ are orthogonal. Suppose that $\vec{u}$ and $\vec{v}$ are nonzero orthogonal vectors. What would $\vec{w}_1$ and $\vec{w}_2$ be in this case?

(b) We want to switch the roles of $\vec{u}$ and $\vec{v}$ for the examples in the previous parts. Specifically, for these configuration of vectors, we want to split $\vec{v}$ into parts that are parallel to $\vec{u}$, which we will call $\vec{z}_1$, and orthogonal to $\vec{u}$, which we will call $\vec{z}_2$. On Figure 9.4.9, draw $\vec{z}_1$ and $\vec{z}_2$ for each configuration.

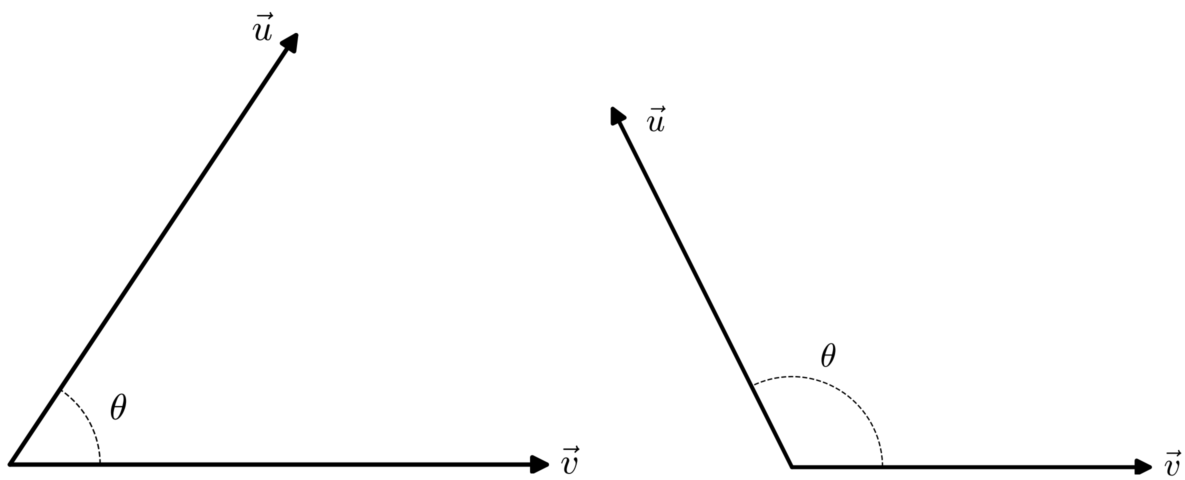


Figure 9.4.9 A plot of $\vec{u}$ and $\vec{v}$

Based on your drawing, is it the case that $\vec{w}_1 = \vec{z}_1$? What about $\vec{w}_2$ and $\vec{z}_2$?

Activity 9.4.6. Suppose that $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$. We wish to find $\vec{w}_1$ and $\vec{w}_2$ so that $\vec{w}_1$ is parallel to $\vec{v}$ and $\vec{w}_2$ is orthogonal to $\vec{v}$, as shown in Figure 9.4.8.

(a) Explain why $\vec{u} = \vec{w}_1 + \vec{w}_2$.

(b) Compute $\vec{u} \cdot \vec{v}$ as $(\vec{w}_1 + \vec{w}_2) \cdot \vec{v}$. Simplify your answer as much as possible, using the fact that $\vec{w}_2$ is orthogonal to $\vec{v}$.

(c) Since $\vec{w}_1$ is parallel to $\vec{v}$, there is a scalar k so that $\vec{w}_1 = k\vec{v}$. Substitute $k\vec{v}$ for $\vec{w}_1$ in your answer to the previous part and then solve for k.

(d) Give a formula for $\vec{w}_1$ purely in terms of $\vec{u}$ and $\vec{v}$.

Activity 9.4.7. Let $\vec{u} = \langle 2, 6 \rangle$.

(a) Let $\vec{v} = \langle 4, -8 \rangle$. Find $\text{proj}_{\vec{v}}\vec{u}$, $\text{comp}_{\vec{v}}\vec{u}$, and $\text{proj}_{\perp\vec{v}}\vec{u}$. Draw a picture to illustrate the vectors involved. Finally, express $\vec{u}$ as the sum of two vectors where one is parallel to $\vec{v}$ and the other is perpendicular to $\vec{v}$.

(b) Now let $\vec{w} = \langle -2, 4 \rangle$. Add $\vec{w}$ to the picture you drew in the previous part. Without doing any calculations, find $\text{proj}_{\vec{w}}\vec{u}$. Explain your reasoning.

(c) Find a vector $\vec{w}$ not parallel to $\vec{z} = \langle 3, 4 \rangle$ such that $\text{proj}_{\vec{z}}\vec{w}$ has length 10. Note that there are infinitely many different answers!

Activity 9.4.8.

(a) Determine the work done by a 25 pound force acting at a 30° angle to the direction of the object's motion, if the object is pulled 10 feet.

(b) Determine if more work or less work is done if the angle to the direction of the object's motion is 60°.

9.5 The Cross Product

Preview Activity 9.5.1.

(a) Our first task is to understand what it means to complete a right-handed coordinate system. For each of the cases below, you need to give a vector that fills in the blank to create a right-handed coordinate system. For example, the answer that would complete $\{\hat{\imath}, \hat{\jmath}, ???\}$ would be $\hat{k}$. You should be careful to when you have vectors that are in the opposite directions of $\hat{\imath}$ or $\hat{\jmath}$ or $\hat{k}$.

(i) $\{\hat{\jmath}, \hat{\imath}, ???\}$

(ii) $\{-\hat{\imath}, \hat{\jmath}, ???\}$

(iii) $\{\hat{k}, \hat{\imath}, ???\}$

(iv) $\{-\hat{\imath}, -\hat{\jmath}, ???\}$

(v) $\{\hat{k}, \hat{\jmath}, ???\}$

(b) Explain why there is not a way to complete $\{\hat{\imath}, \hat{\imath}, ???\}$ to be a right-handed coordinate system.

(c) As we said above, we would like the cross product to distribute over vector sums $((\vec{v} + \vec{u}) \times \vec{w} = (\vec{v} \times \vec{w}) + (\vec{u} \times \vec{w}))$. Complete each of the following right-handed coordinate systems geometrically. Focus on the direction of the vector needed to complete the right-handed coordinate system and do not worry about the length of the vector you choose.

(i) $\{\hat{\imath}, \hat{k}, ???\}$

(ii) $\{\hat{\jmath}, \hat{k}, ???\}$

(iii) $\{\hat{\imath} + \hat{\jmath}, \hat{k}, ???\}$

Be sure to note how your thumb is oriented in the last case and verify that the sum of the first two results gives you the last case.

(d) Finally, we want to address the relationship between the cross product and scalar multiplication.

(i) What direction should the third vector in the right-handed coordinate system $\{2\hat{\imath}, 3\hat{\jmath}, ???\}$ be?

(ii) What direction should the third vector in the right-handed coordinate system $\{2\hat{k}, \hat{\jmath}, ???\}$ be?

(iii) Note that $3\hat{k} = \hat{k} + \hat{k} + \hat{k}$, so $(3\hat{k}) \times \hat{\jmath} = (\hat{k} \times \hat{\jmath}) + (\hat{k} \times \hat{\jmath}) + (\hat{k} \times \hat{\jmath})$. What do you think the vector $(3\hat{k}) \times \hat{\jmath}$ will be?

(iv) What do you think the result of $(4\hat{\imath}) \times (-6\hat{\jmath})$ will be?

Activity 9.5.2. Suppose $\vec{u} = \langle 0, 1, 3\rangle$ and $\vec{v} = \langle 2, -1, 0\rangle$. Use equation (9.5.1) for the following.

(a) Find the cross product $\vec{u} \times \vec{v}$.

(b) Evaluate the dot products $\vec{u} \cdot (\vec{u} \times \vec{v})$ and $\vec{v} \cdot (\vec{u} \times \vec{v})$. What does this tell you about the geometric relationship among $\vec{u}$, $\vec{v}$, and $\vec{u} \times \vec{v}$?

(c) Find the cross product $\vec{v} \times \hat{\imath}$.

(d) Recall that multiplication of real numbers is *associative*. For example, $(2 \cdot 5) \cdot 3 = 2 \cdot (5 \cdot 3)$. Is it true that the cross product of vectors is associative? For instance, is it true that $(\vec{u} \times \vec{v}) \times \hat{\imath} = \vec{u} \times (\vec{v} \times \hat{\imath})$?

(e) Find the cross product $\vec{u} \times \vec{u}$ and write a sentence or two to explain the meaning of your result.

Activity 9.5.3.

(a) Find the area of the parallelogram formed by the vectors $\vec{u} = \langle 1, 3, -2 \rangle$ and $\vec{v} = \langle 3, 0, 1 \rangle$.

(b) Find the area of the parallelogram in $\mathbb{R}^3$ whose vertices are $(1, 0, 1)$, $(0, 0, 1)$, $(2, 1, 0)$, and $(1, 1, 0)$.

Activity 9.5.4. Suppose $\vec{u} = \langle 3, 5, -1 \rangle$ and $\vec{v} = \langle 2, -2, 1 \rangle$.

(a) Find two unit vectors orthogonal to both $\vec{u}$ and $\vec{v}$.

(b) Find the volume of the parallelepiped formed by the vectors $\vec{u}$, $\vec{v}$, and $\vec{w} = \langle 3, 3, 1 \rangle$.

(c) Find a vector orthogonal to the parallelogram containing the points $(0, 1, 2)$, $(4, 1, 0)$, and $(-2, 2, 2)$.

(d) Given the vectors $\vec{u}$ and $\vec{v}$ shown below in Figure 9.5.10, sketch the cross products $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$.

A plot with three mutually perpendicular axes labeled with x going down and left, y going to the right and z going up. The first black arrow is shown going up and to the right and is labeled $\vec{v}$. A second black arrow going to the left and up and is labeled $\vec{u}$.

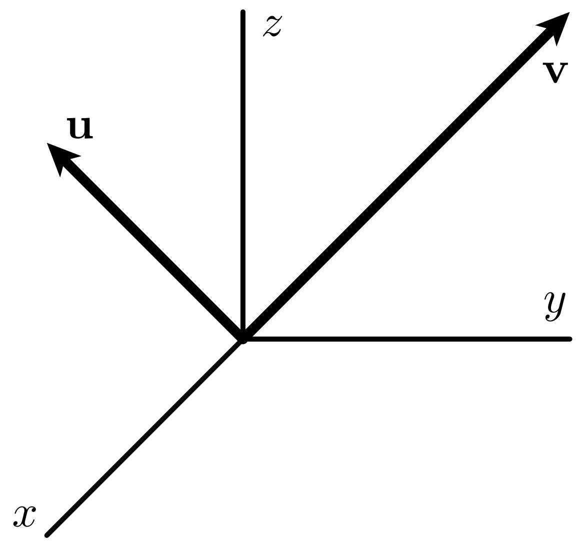


Figure 9.5.10 Vectors $\vec{u}$ and $\vec{v}$

(e) Are the vectors $\vec{a} = \langle 1, 3, -2 \rangle$,$\vec{b} = \langle 2, 1, -4 \rangle$, and $\vec{c} = \langle 0, 1, 0 \rangle$ in standard position coplanar? Use the concepts from this section to explain your answer.

Activity 9.5.5. In this activity, we are focused on the type of objects being used and whether the expression makes sense to do at all. We are *not* going to worry about interpreting or understanding what is being measured by these expressions.

(a) For each of the expressions below, state whether the result is a scalar, a vector, or undefined. You should write a sentence or two about each to explain your reasoning. (Assume that vectors are nonzero and not orthogonal.)

(i) $\dfrac{\vec{v}}{\vec{w}}$

(ii) $\dfrac{\vec{v}}{\vec{w}\cdot\vec{v}}$

(iii) $(\vec{u}\times\vec{w})+\vec{v}$

(iv) $k(\vec{u}\cdot\vec{w})+c\vec{v}$

(v) $k(\vec{u}\cdot\vec{v})\cdot\vec{w}$

(vi) $\dfrac{\vec{v}}{\|\vec{w}\|}$

(vii) $(k+\vec{u})\times\vec{v}$

(viii) $k+(\vec{u}\times\vec{v})$

(ix) $\vec{v}+(\vec{u}\cdot\vec{w})$

(x) $k+\|\vec{u}\times\vec{w}\|$

(b) Use the operations of dot product and vector subtraction to write an expression involving $\vec{u}$, $\vec{v}$, and $\vec{w}$ that evaluates to a scalar. You can use other operations if you want.

(c) Use the operations of cross product, scalar multiplication, and vector addition to write an expression involving $\vec{u}$, $\vec{v}$, and $\vec{w}$ that evaluates to a vector. You can use other operations if you want.

(d) Use the operations of dot product and vector addition to write an expression involving $\vec{u}$, $\vec{v}$, and $\vec{w}$ that is undefined. You can use other operations if you want.

9.6 Lines in Space

Preview Activity 9.6.1. We will start our work on lines by considering some familiar ideas in $\mathbb{R}^2$ but from a new perspective. You are probably familiar with equations of lines in the xy-plane in the form $y = mx + b$, where m is the slope of the line and $(0, b)$ is the y-intercept. In this activity, we explore a more flexible way of representing lines that is useful in the xy-plane and higher dimensions. To begin, consider the line through the point $(2, -1)$ with slope $\frac{2}{3}$ as shown in Figure 9.6.2.

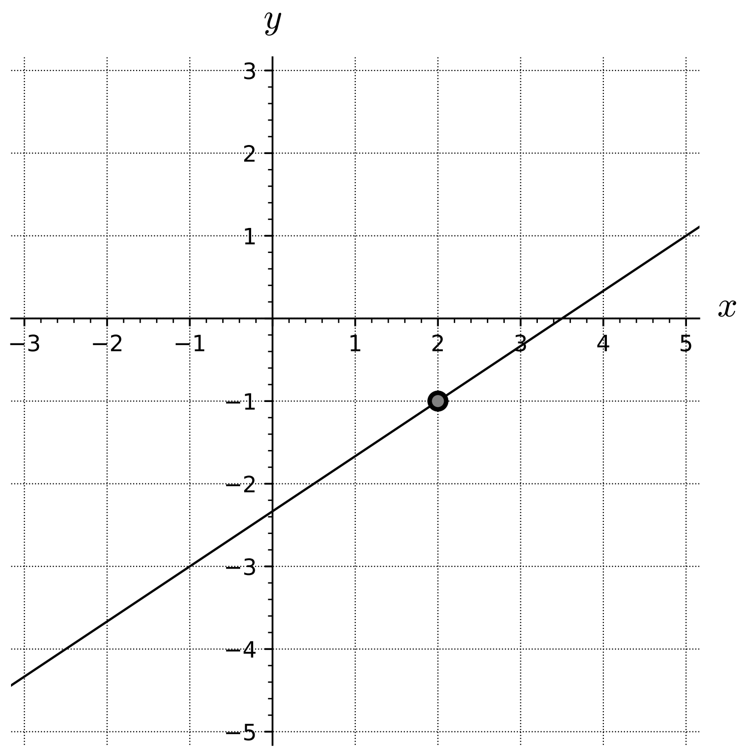


Figure 9.6.2 The line through $(2, -1)$ with slope $\frac{2}{3}$

(a) Suppose we increase x by 1 from the point $(2, -1)$. How does the y-value of a point on the line change? What is the point on the line with x-coordinate 3?

(b) Suppose we decrease x by 3.25 from the point $(2, -1)$. How does the y-value change? What is the point on the line with x-coordinate -1.25?

(c) Now, suppose we increase x by some arbitrary value $3t$ from the point $(2,-1)$. How does the y-value change? What is the point on the line with x-coordinate $2+3t$?

(d) Remember that the horizontal component of a vector describes the "run" and the vertical component measures the "rise" of the vector. So the slope of the line is related to *any* vector whose y-component divided by the x-component is the slope of the line. For the line in this activity, we might use the vector $\langle 3,2\rangle$ to describe the direction of the line. We will look at the following vector-valued function of one variable:

$$\vec{r}(t) = \langle 2,-1\rangle + \langle 3,2\rangle t,$$

For each of the values of t below, find $\vec{r}(t)$ and write your result as a single vector (of the form $\langle a,b\rangle$)

(a) $t=0$

(b) $t=1$

(c) $t=2$

(d) $t=-2$

(e) $t=-\dfrac{1}{2}$

(f) $t=\dfrac{7}{3}$

(e) Draw the six vectors from the previous step in standard position (with initial point at the origin) on the plot below.

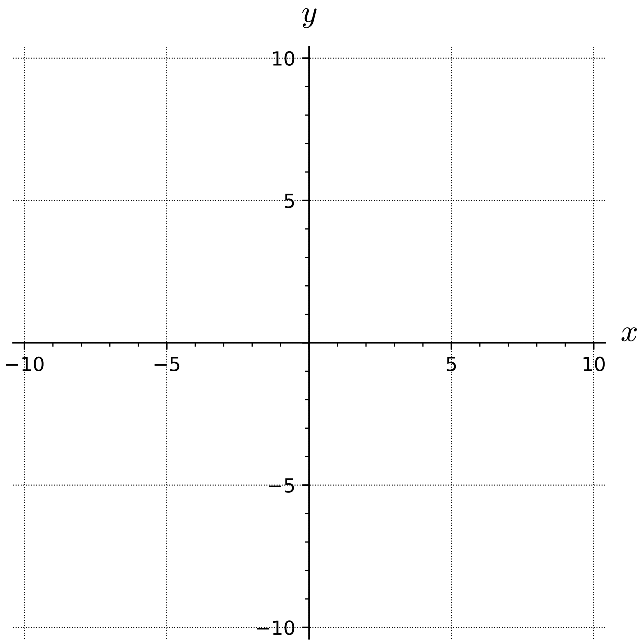


Figure 9.6.3 A blank 2D set of axes

(f) Write a few sentences that compare the endpoints of your vectors from the previous task to the plot of the line through the point $(2, -1)$ with a slope of $\frac{2}{3}$ (Figure 9.6.2). In particular you should address what aspects of the line are related to $\langle 2, -1\rangle$ or $\langle 3, 2\rangle$.

Activity 9.6.2. Let $P_1 = (1, 2, -1)$ and $P_2 = (-2, 1, -2)$ and let $\mathcal{L}$ be the line in $\mathbb{R}^3$ through P_1 and P_2. Note that Figure 9.6.9 shows a similar example of a line in 3D defined by two points.

(a) Find a direction vector for the line $\mathcal{L}$.

(b) Find a vector equation of $\mathcal{L}$ in the form $\vec{r}(t) = \vec{r}_0 + t\vec{v}$.

(c) Consider the vector equation $\vec{s}(t) = \langle -5, 0, -3\rangle + t\langle 6, 2, 2\rangle$. What is the direction of the line given by $\vec{s}(t)$? Is this new line parallel to line $\mathcal{L}$?

(d) Do $\vec{r}(t)$ and $\vec{s}(t)$ represent the same line, $\mathcal{L}$? Write a couple of sentences to justify why you think $\vec{r}(t)$ and $\vec{s}(t)$ do or do not describe the same set of points.

Activity 9.6.3. Let $P_1 = (1, 2, -1)$ and $P_2 = (-2, 1, -2)$, and let $\mathcal{L}$ be the line in $\mathbb{R}^3$ through P_1 and P_2. (This is the same line as in Activity 9.6.2.)

(a) Find parametric equations of the line $\mathcal{L}$.

(b) Does the point $(1, 2, 1)$ lie on $\mathcal{L}$? If so, what value of t results in this point?

(c) Consider another line, $\mathcal{K}$, whose parametric equations are

$$x(s) = 11 + 4s, \quad y(s) = 1 - 3s, \quad z(s) = 3 + 2s.$$

What is the direction of the line $\mathcal{K}$?

(d) Do the lines $\mathcal{L}$ and $\mathcal{K}$ intersect? If so, provide the point of intersection and the t and s values, respectively, that result in the point. If not, explain why. To find a point of intersection, you can set the coordinate equations of each line equal to each other try to solve for t and s.

9.7 Planes in Space

Preview Activity 9.7.1. We will consider what happens when we allow movement in $\mathbb{R}^3$ with the restriction that the movement must be orthogonal to $\vec{v} = \langle 1, 2, 3\rangle$.

(a) Find values for a_0 and b_0 such that $\langle 0, a_0, b_0\rangle$ is orthogonal to $\langle 1, 2, 3\rangle$.

(b) Find values for c_0 and d_0 such that $\langle c_0, d_0, 0\rangle$ is orthogonal to $\langle 1, 2, 3\rangle$.

(c) Find values for c_1 and d_1 such that $\langle c_1, d_1, 1\rangle$ is orthogonal to $\langle 1, 2, 3\rangle$.

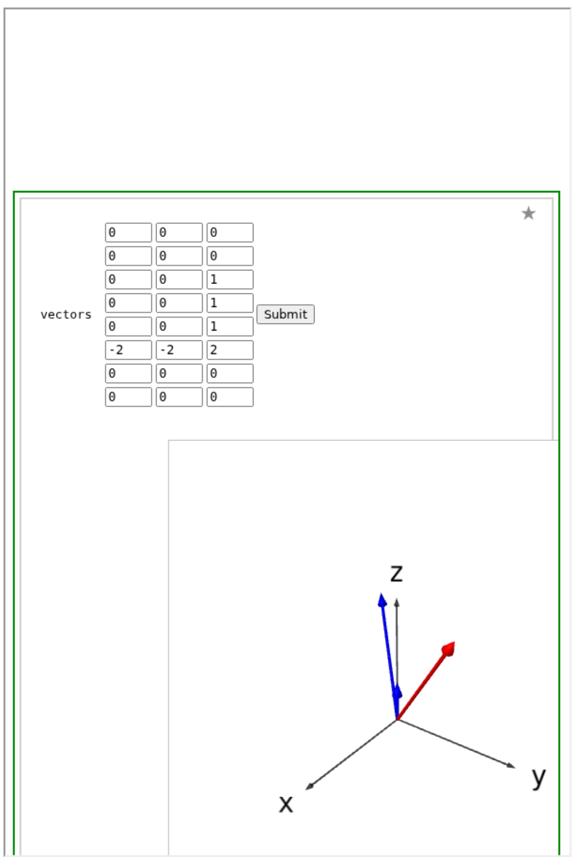
(d) Find two other values for each of c and d such that $\langle c, d, 1\rangle$ is orthogonal to $\langle 1, 2, 3\rangle$.

(e) Verify that each of the following vectors is also orthogonal to $\langle 1, 2, 3\rangle$.

- $\langle -2, -2, 2\rangle$
- $\langle c_0, a_0 + d_0, b_0\rangle$ with your values from parts a and b

- $\langle -2c_1, a_0 - 2d_1, b_0 - 2\rangle$ with your values from parts a and c

(f) Put all of the vectors you have computed into the interactive below to visually verify that each of them is orthogonal to $\langle 1, 2, 3\rangle$. You should put the component values of each vector into this array with each vector corresponding to a row. If you do not have eight distinct vectors from the previous parts, multiply one of your repeated vectors by -1 and enter that instead of entering a vector multiple times.



Standalone

In Context

Embed

Figure 9.7.1 A plot of your vectors that should be orthogonal to $\langle 1, 2, 3\rangle$

(g) Describe what you think the plot of the set of vectors that are orthogonal to $\langle 1, 2, 3\rangle$ will look like.

Activity 9.7.2.

(a) Write a scalar equation of the plane p_1 passing through the point $(0, 2, 4)$ and perpendicular to the vector $\vec{n} = \langle 2, -1, 1 \rangle$.

(b) Is the point $(2, 0, 2)$ on the plane p_1?

(c) Write a scalar equation of the plane p_2 that is parallel to p_1 and passes through the point $(3, 0, 4)$.

(d) Give parametric equations for the line $\mathcal{L}$ passing through the point $(2, 0, 2)$ and perpendicular to the plane p_3 described by the equation $x + 2y - 2z = 7$.

(e) Find the point at which $\mathcal{L}$ intersects p_3.

Activity 9.7.3. Let $P_0 = (1, 2, -1)$, $P_1 = (1, 0, -1)$, and $P_2 = (0, 1, 3)$ and let p be the plane containing P_0, P_1, and P_2.

(a) Determine the components of the vectors $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$.

(b) Find a normal vector $\vec{n}$ to p.

(c) Find a scalar equation of p.

(d) Consider a second plane q with scalar equation $-3(x-1) + 4(y+3) + 2(z-5) = 0$. Find two different points on q as well as a vector $\vec{m}$ that is normal to q.

(e) What is the angle between planes p and q?

9.8 Common Graphs in Three Dimensions

Preview Activity 9.8.1. For each equation below, identify its graph in Figure 9.8.1. Note that there are more graphs than equations, so some graphs will not be selected. To help you with identification, you might consider looking for x-intercepts, y-intercepts, and values of a variable for which the graph contains no points.

i. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

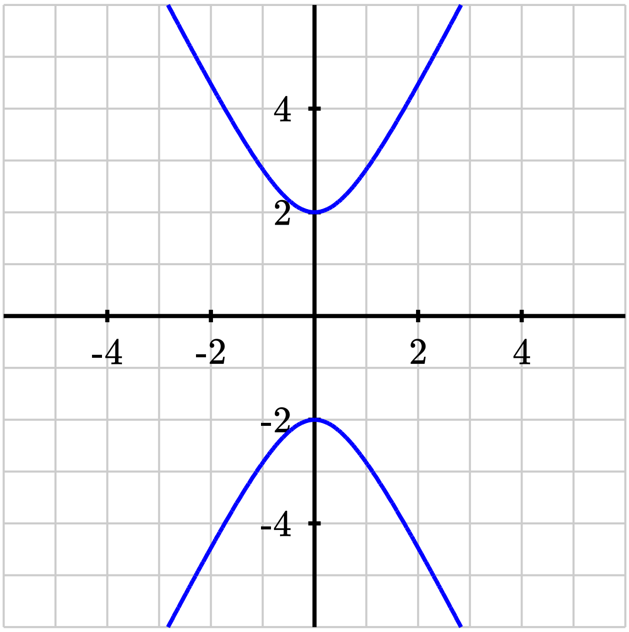
ii. $\dfrac{x}{3} + \dfrac{y}{5} = 1$

iii. $\dfrac{y^2}{4} - \dfrac{x^2}{1} = 1$

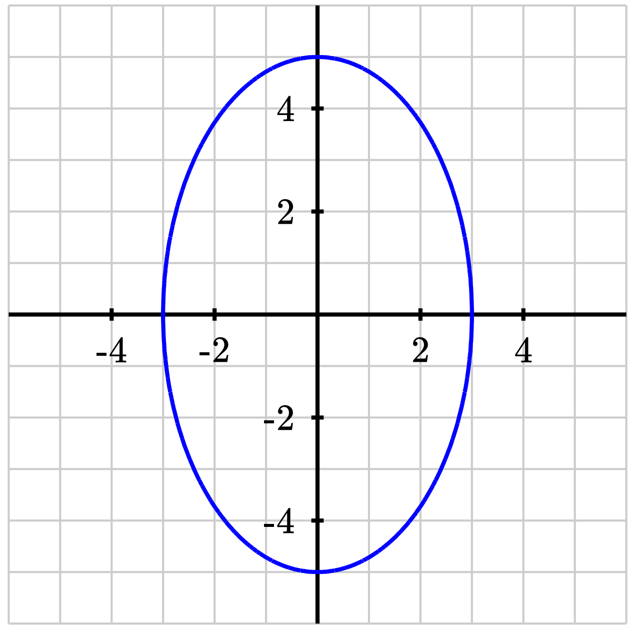
iv. $x = 2y^2$

v. $\dfrac{y^2}{9} - \dfrac{x^2}{25} = 0$

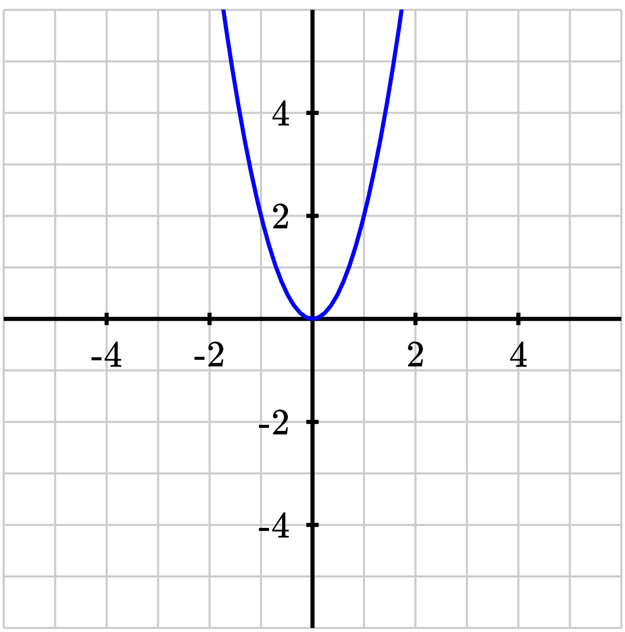
vi. $\dfrac{x^2}{1} - \dfrac{y^2}{4} = 1$



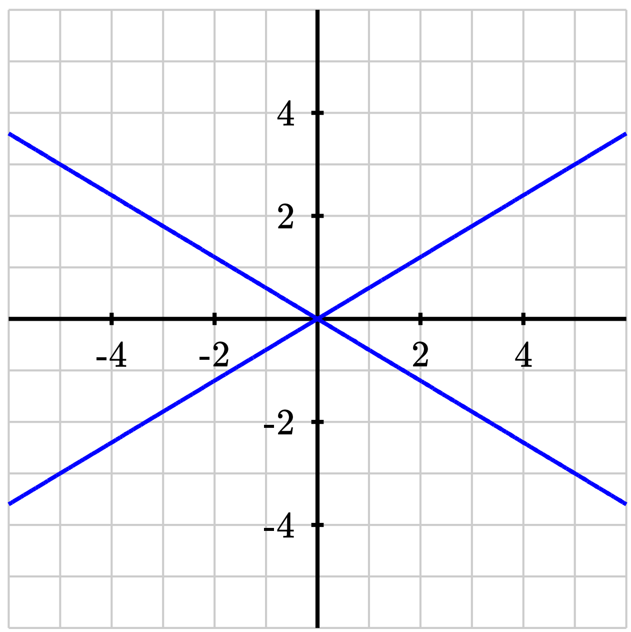
(a)



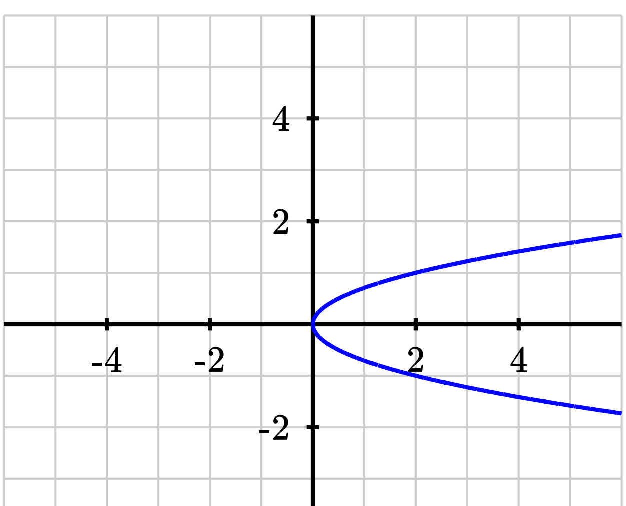
(b)

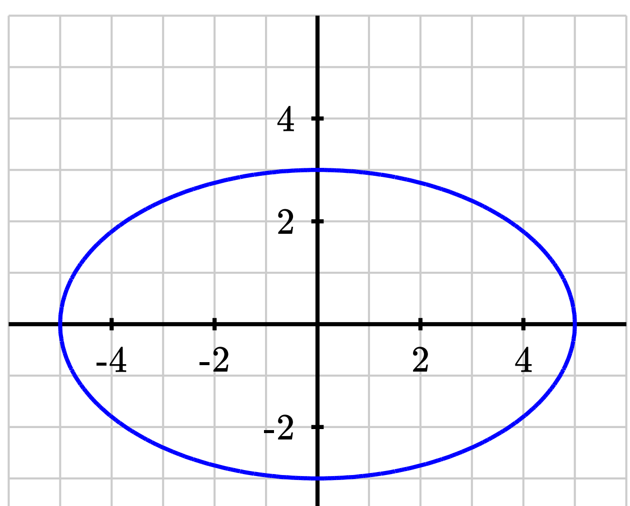


(c)



(d)





Activity 9.8.2. Each equation below is a cylinder surface in $\mathbb{R}^3$. To sketch the cylinder surfaces in xyz-space, you should first draw the generating curve in the xy-plane, xz-plane, or yz-plane (depending on which two variables appear in the equation) and then sketch a three-dimensional cylinder surface by thinking about how the rulings will run.

(a) $2x - y + 1 = 0$ is called a **linear cylinder surface**.

(b) $(x-1)^2 + (y+2)^2 = 4$ is called a **right-circular cylinder surface**.

(c) $\frac{x^2}{9} + \frac{z^2}{4} = 1$ is called an **elliptic cylinder surface**.

(d) $x^2 - y^2 = 1$ is called a **hyperbolic cylinder surface**.

Activity 9.8.3. For this activity, we will be looking at a variety properties that will help us draw a graph of the surface described by $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$.

(a) Find all x-, y-, and z-intercepts of $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$.

(b) Find an equation for the curve given by the intersection of $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$ with the xy-plane, the yz-plane, and the xz-plane. Draw a two-dimensional plot of each intersection.

(c) Find equations for the curve given by the intersection of $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$ with the each of the following fundamental planes. You should state the shape and any other characteristics (like center or direction) for each of these intersections.

(i) $z = \sqrt{3}$

(ii) $z = -2$

(iii) $x = 3$

(iv) $x = -1$

(v) $y = 2$

(vi) $y = -4$

(d) Sketch each of these intersections on the proper fundamental planes in three dimensions.

(e) Which of the following surface plots will correspond to $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$? You can determine this by comparing the features on your previous part to these options.

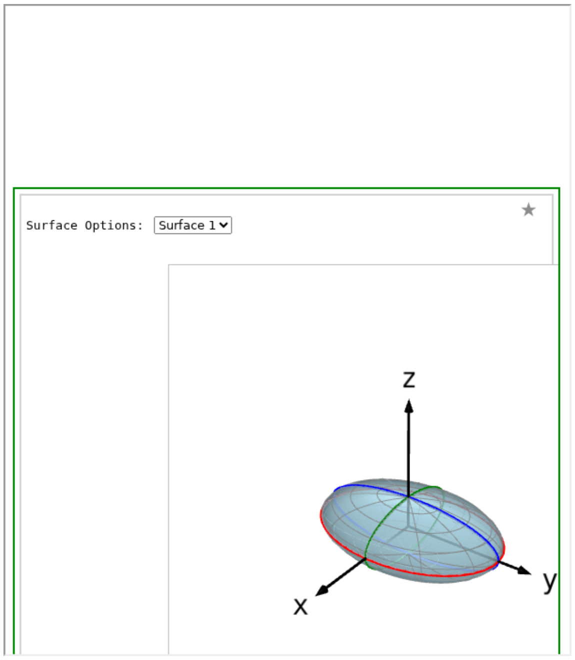


Figure 9.8.10 Choose the surface with equation $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$

Standalone

In Context

Embed

Activity 9.8.4. For each equation below, use the process in Key Idea 9.8.11 to determine the shape of the quadric surface defined by the equation and identify the graph of the quadric surface in Figure 9.8.12.

(a) $\dfrac{x^2}{4} - \dfrac{y^2}{9} - \dfrac{z^2}{1} = 1$

(b) $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{1} = 1$

(c) $\dfrac{x^2}{4} - \dfrac{y^2}{9} - \dfrac{z}{1} = 0$

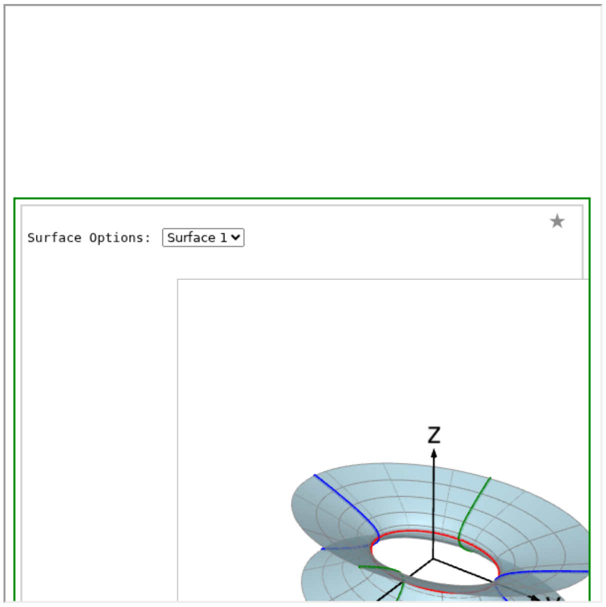


Figure 9.8.12 A plot of surfaces to select from

Standalone

In Context

Embed

9.9 Polar, Cylindrical, and Spherical Coordinates

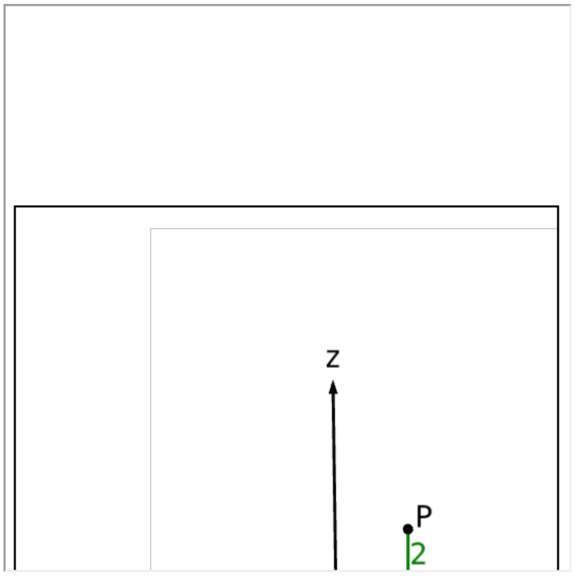
Preview Activity 9.9.1. An angle in the xy-plane is in standard position if the initial side of the angle is on the positive x-axis and is measured with the positive direction going counterclockwise.

(a) If an angle θ is in standard position, for which quadrants will $\sin(\theta)$ be positive? For which quadrants will $\sin(\theta)$ be negative? Which angles correspond to $\sin(\theta) = 0$?

(b) If an angle θ is in standard position, for which quadrants will $\cos(\theta)$ be positive? For which quadrants will $\cos(\theta)$ be negative? Which angles correspond to $\cos(\theta) = 0$?

(c) If an angle θ is in standard position, for which quadrants will $\tan(\theta)$ be positive? For which quadrants will $\tan(\theta)$ be negative? Which angles correspond to $\tan(\theta) = 0$? Which angles correspond to $\tan(\theta)$ being undefined?

(d) Consider the plot of point P in Figure 9.9.1.



Standalone

In Context

Embed

Figure 9.9.1 A plot of point P with y, z, and r labeled

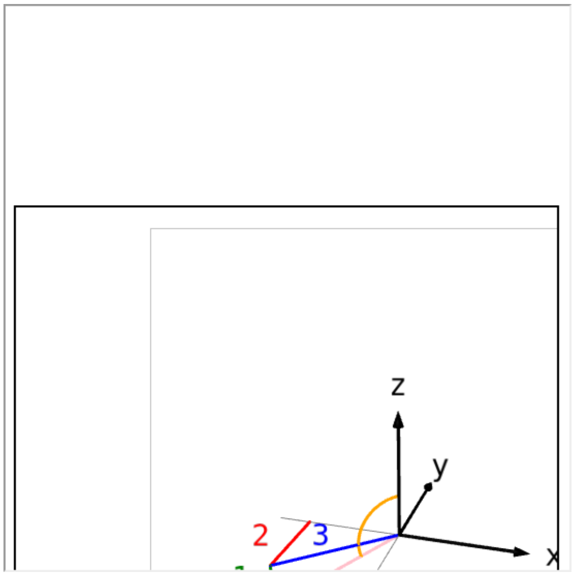
Use trigonometry and the distance formula to find the following:

(i) The x-coordinate of P.

(ii) The angle in the xy-plane between the x-axis and the line segment (in blue) labeled 5 that is from the origin to a point directly below P.

(iii) The distance between the origin and P.

(e) Consider the plot of point P in Figure 9.9.2.



Standalone

In Context

Embed

Figure 9.9.2 A plot of point P with y, z, and r labeled

Use trigonometry and the distance formula to find the following:

(i) The x-coordinate of P.

(ii) The angle between the z-axis and the pink line segment from the origin to P.

(iii) The distance between the origin and P.

Activity 9.9.2.

(a) For each of the points listed below, you should:

- Graph the point on a set of axes and label how each of the rectangular coordinates is measured.
- Draw and label how the polar coordinates are measured for each point.
- Compute the exact value of r and θ for each point. Exact values include things like $\sqrt{3}$, $\arcsin(3/4)$, etc. Simplify trigonometric function values of any common angles you encounter, like $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

(i) $(x, y) = (-3, -7)$

(ii) $(x, y) = (7, -3)$

(iii) $(x, y) = (\sqrt{5}, -2)$

(iv) $(x, y) = \left(\frac{3}{2}, -\frac{3}{2}\right)$

(v) $(x, y) = \left(\frac{3}{2}, \frac{3}{2}\right)$

(b) For each of the points listed below, you should:

- Graph the point on a set of axes and label how each of the polar coordinates is measured.
- Draw and label how the rectangular coordinates are measured for each point.
- Compute the exact value of x and y for each point. Exact values include things like $\sqrt{3}$, $\arcsin(3/4)$, etc. Simplify trigonometric function values of any common angles you encounter, like $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

(i) $(r, \theta) = \left(1, -\frac{\pi}{6}\right)$

(ii) $(r, \theta) = \left(\sqrt{3}, -\arctan\left(\frac{\sqrt{2}}{2}\right)\right)$

(iii) $(r, \theta) = \left(-\sqrt{5}, 0\right)$

(iv) $(r, \theta) = \left(7, \frac{3\pi}{4}\right)$

(v) $(r, \theta) = \left(3, \pi - \frac{\pi}{6}\right)$

Activity 9.9.3. This activity gives you an opportunity to graph some equations in polar coordinates where r is expressed as a function of θ.

(a) The following activity will take you through how to create a graph of an equation in r and θ.

(i) Draw a graph of $y = \cos(x)$ for at least $0 \leq x \leq 2\pi$.

(ii) Consider the polar equation $r = \cos(\theta)$. Complete the table below by computing the value of r for each value of θ for this equation.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r											

(iii) Plot each of the eleven points you found in the previous part on the polar plane and connect them to make a plot of the graph of $r = \cos(\theta)$.

(b) Graph the equation $r = 1 + \cos(\theta)$ in the polar plane.

(c) Graph the equation $r = 1 - \cos(\theta)$ in the polar plane.

(d) What are the points of intersections for the graphs of $r = 1 + \cos(\theta)$ and $r = 1 - \cos(\theta)$

Activity 9.9.4. Below are several equations, some in polar form and some in rectangular form. For each equation, complete the following steps:

i. Convert the equation to the other coordinate system (either rectangular to polar or polar to rectangular).

ii. State the shape of the graph for the equation and any other information needed to graph

iii. Graph the given equation and write a sentence explaining how the graph helps understand the converted equation you found in the first step.

(a) $r = 2$

(b) $\theta = \frac{\pi}{3}$

(c) $x = -1$

(d) $y = 2$

(e) $y = -x$

(f) $r = \dfrac{3}{1 - 2\cos(\theta)}$

Activity 9.9.5.

(a) Write a couple of sentences describing the graph of $r = c$, where c is a constant. You should consider what values of c will correspond to different shapes.

(b) Write a couple of sentences describing the graph of $\theta = d$, where d is a constant. You should consider what values of d will correspond to different shapes.

Activity 9.9.6.

(a) Graph the points that satisfy $y = x + 1$.

(b) On the same axes as in the previous part, graph the points that satisfy $y \geq x + 1$.

(c) Graph the points that satisfy the inequality $x^2 + y^2 \leq 4$.

(d) Graph the points that satisfy both $x^2 + y^2 \leq 4$ and $y \geq x + 1$.

Activity 9.9.7.

(a) Plot the region that corresponds to the polar inequality $1 \leq r \leq 3$. Write a sentence or two to describe this region.

(b) Plot the region that corresponds to the polar inequality $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$. Write a sentence or two to describe this region.

(c) Plot the region that corresponds to the polar inequalities $1 \leq r \leq 3$ and $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$. Write a sentence or two to describe this region.

(d) Plot the region that corresponds to the polar inequality $1 \leq r \leq 3\cos(\theta)$. Write a sentence or two to describe this region.

Activity 9.9.8.

(a) Convert $(x, y, z) = (1, 2, 3)$ to cylindrical coordinates.

(b) Use inequalities on cylindrical coordinates to describe the region given by the 6th octant.

(c) Convert the cone $z^2 = x^2 + y^2$ to cylindrical coordinates. Write a couple of sentences to make sense of how you can simplify your conversion and describe the shape of the graph in terms of z and r.

(d) Draw a plot of the surface in $\mathbb{R}^3$ with equation $r = 2$ in cylindrical coordinates. Write a couple of sentences about the shape and properties of this surface.

(e) Draw a plot of the region given by $0 \le \theta \le \pi, 0 \le r \le 2, 0 \le z \le r^2$. Write a couple of sentences about the shape and properties of this region.

Activity 9.9.9. In this activity, we graph some surfaces using cylindrical coordinates. To improve your intuition and test your understanding, you should first think about what each graph should look like before you plot it using appropriate technology.

(a) What familiar surface is described by the points in cylindrical coordinates with $r = 2$, $0 \leq \theta \leq 2\pi$, and $0 \leq z \leq 2$? How does this example suggest that we call these coordinates *cylindrical coordinates*? How does the surface change if we restrict θ to $0 \leq \theta \leq \pi$?

(b) What familiar surface is described by the points in cylindrical coordinates with $\theta = 2$, $0 \leq r \leq 2$, and $0 \leq z \leq 2$?

(c) What familiar surface is described by the points in cylindrical coordinates with $z = 2$, $0 \leq \theta \leq 2\pi$, and $0 \leq r \leq 2$?

(d) Plot the graph of the cylindrical equation $z = r$, where $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. What familiar surface results?

(e) Plot the graph of the cylindrical equation $z = \theta$ for $0 \leq \theta \leq 4\pi$. What does this surface look like?

Activity 9.9.10.

(a) Use the conversion equations between rectangular and spherical coordinates to find the rectangular coordinates of the following points:

(i) $(\rho, \theta, \phi) = (3, \frac{\pi}{3}, \frac{\pi}{2})$

(ii) $(\rho, \theta, \phi) = (4, \frac{4\pi}{3}, \frac{3\pi}{4})$

(iii) $(\rho, \theta, \phi) = (\sqrt{7}, 20^\circ, \pi)$

(b) Draw each of the points from part a and show how the spherical coordinates of each point is being measured. You should use your plots to make sense of the rectangular coordinate measurements that were your answer to part a.

Activity 9.9.11. For many students, the measurement of ϕ is the most difficult and most unfamiliar measurement for spherical coordinates. In order to help understand the measurement of the ϕ-coordinate, we will look at the collection of points described by a constant value of ϕ. For each of the equations below, you should:

i. Draw a plot of the points that satisfy that equation. For some equations, your plot will be a path in space and others will correspond to a surface. You should draw how the ϕ-coordinate is measured and related to the graph.

ii. Write a sentence or two that describes the shape and features of your plot of the equation. You should mention any other descriptions that could be applied to your set of points, such as axes or coordinate planes.

(a) $\phi = 0$

(b) $\phi = 1$

(c) $\phi = \frac{\pi}{2}$

(d) $\phi = \frac{3\pi}{4}$

(e) $\phi = \pi$

Activity 9.9.12. For each of the surfaces described below, do the following:

i. Find a corresponding equation in spherical coordinates (involving only ρ, θ, and ϕ). You may want to consider algebraic or geometric approaches to these problems.

ii. Draw a plot of surface and label any important measurements in terms of the spherical coordinates.

iii. Write a sentence or two about the corresponding equation in spherical coordinates and any assumptions made in your work.

(a) the xy-plane

(b) the xz-plane

(c) $z = x^2 + y^2$

Activity 9.9.13.

(a) Draw the region described by $\rho \leq 1$ and $0 \leq \phi \leq \frac{\pi}{2}$.

(b) Draw the region described by $x^2 + y^2 \leq 9$ with $x \leq y$ and $-1 \leq z \leq 3$

(c) Give a set of inequalities in spherical coordinates that describes the points in the seventh octant.

10 Vector-Valued Functions of One Variable

10.2 Vector-Valued Functions of One Variable

Preview Activity 10.2.1. After graduating, you decide to start a self-driving car company because it didn't look too hard based on a few articles you scrolled past online. You decide to call your company *Steer Clear* and start working on the self-driving part of the car. Your first task is to understand how the location tracking equipment you bought online works. There are two parts to your location tracking system: a receiver box and a tracker that you put in the object you want to track.

(a) After putting the tracker in your pocket, you get in a car and drive around the receiver box. When you download the data about your drive, you notice that the software is outputting a vector from the receiver box to the tracker. The software also seems to love math, because these vectors are written in terms of trig functions.

Below are some of the output vectors from the software. Evaluate each of these vectors and draw these vectors with initial point at the origin on the axes in Figure 10.2.2.

- $\langle \cos(0), \sin(0) \rangle$
- $\langle \cos\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right) \rangle$
- $\langle \cos(\pi), \sin(\pi) \rangle$
- $\langle \cos\left(\frac{3\pi}{2}\right), \sin\left(\frac{3\pi}{2}\right) \rangle$

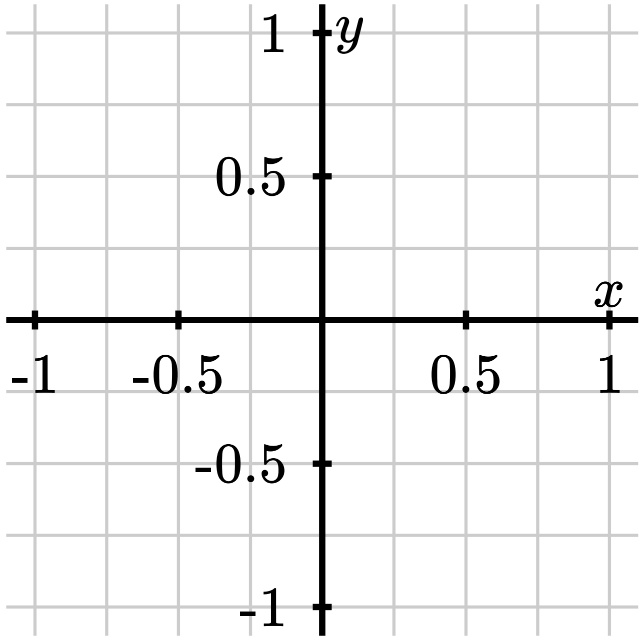


Figure 10.2.2 Two-dimensional axes for plotting points

(b) Below are a few more output vectors from your software. Evaluate each of these vectors and draw each of these vectors with initial point at the origin on the axes in Figure 10.2.2.

- $\langle \cos\left(\frac{\pi}{4}\right), \sin\left(\frac{\pi}{4}\right)\rangle$
- $\langle \cos\left(\frac{3\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right)\rangle$
- $\langle \cos\left(\frac{5\pi}{4}\right), \sin\left(\frac{5\pi}{4}\right)\rangle$
- $\langle \cos\left(\frac{7\pi}{4}\right), \sin\left(\frac{7\pi}{4}\right)\rangle$

(c) Based on the data you are seeing from the two sets of vector outputs of your software, it appears that the output of your software is the set of vectors of the form $\langle \cos(t), \sin(t) \rangle$, where t goes from 0 to 2π. Add to Figure 10.2.2 a sketch the set of *terminal* points (only) of vectors $\langle \cos(t), \sin(t) \rangle$ plotted in standard position (with initial point at the origin). You should allow t to assume values from 0 to 2π. Write a couple sentences about what path your drive took.

(d) What part of the path you drove is described by $\langle \cos(t), \sin(t) \rangle$, where t goes from 0 to π? What would the path be if you drove along $\langle \cos(t), \sin(t) \rangle$ for t from 0 to 4π?

Activity 10.2.2. The same curve can be represented with different parameterizations. Use appropriate technology to plot each of the curves generated by the following vector-valued functions for values of t from 0 to 2π. For each example, you should write a few sentences to explain how the graphs are alike and how they are different. Be sure to pay attention to how the parameter values are related to different points on the curve.

(a) $\vec{r}(t) = \langle \sin(t), \cos(t) \rangle$

(b) $\vec{r}(t) = \langle \sin(2t), \cos(2t) \rangle$

(c) $\vec{r}(t) = \langle \cos(t+\pi), \sin(t+\pi) \rangle$

(d) $\vec{r}(t) = \langle \cos(t^2), \sin(t^2) \rangle$

Activity 10.2.3. Vector-valued functions can be used to generate many interesting curves. Graph each of the following using an appropriate technological tool, and then write one sentence for each function to describe the behavior of the resulting curve.

(a) $\vec{r}(t) = \langle t\cos(t), t\sin(t)\rangle$

(b) $\vec{r}(t) = \langle \sin(t)\cos(t), t\sin(t)\rangle$

(c) $\vec{r}(t) = \langle \sin(5t), \sin(4t)\rangle$

(d) $\vec{r}(t) = \langle t^2\sin(t)\cos(t), 0.9t\cos(t^2), \sin(t)\rangle$. (Note that this defines a curve in three-dimensional space.)

Experiment with different formulas for $x(t)$ and $y(t)$ and ranges for t to see what other interesting curves you can generate. Share your best results with peers.

Activity 10.2.4. For this activity we will consider the paraboloid with equation $z = x^2 + y^2$. You do not need to know what the graph of $z = x^2 + y^2$ looks like at this point.

(a) Parameterize the intersection of the paraboloid with the fundamental plane $x = 2$. What type of curve is the intersection?

(b) Parameterize the intersection of the paraboloid with the fundamental plane $y = -1$. What type of curve is the intersection?

(c) Parameterize the intersection of the paraboloid with the fundamental plane $z = 25$. What type of curve is the intersection?

(d) How do your responses change to all three of the preceding questions if you instead consider the surface defined by $z = x^2 - y^2$?

10.3 Calculus of Vector-Valued Functions of One Variable

Preview Activity 10.3.1. As the only employee of *Steer Clear*, you have decided that you need to understand how the timing of position measurements will change different properties related to your self-driving car. You decide to drive in a figure eight path described by $\vec{r}_8(t) = \langle \cos(t), \sin(2t) \rangle$ for $0 \leq t \leq 2\pi$. A plot of this path is given in Figure 10.3.1.

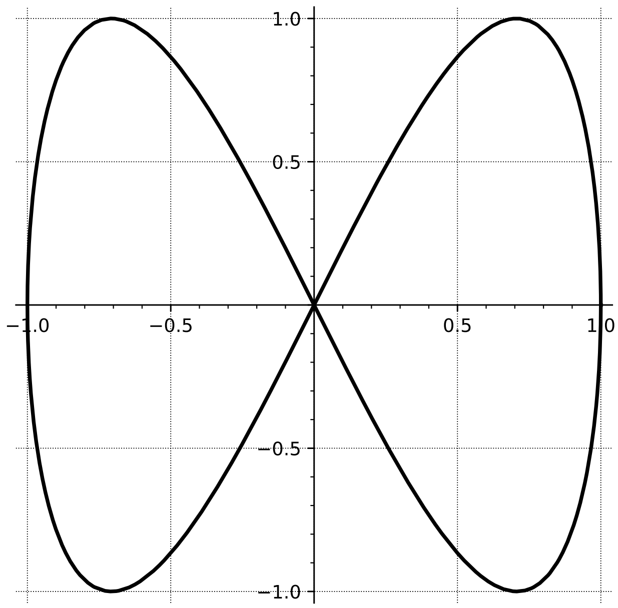


Figure 10.3.1 A plot of the figure eight curve given by $\vec{r}_8(t) = \langle \cos(t), \sin(2t) \rangle$ for $0 \leq t \leq 2\pi$.

(a) In order to understand how often your software should collect location data, you decide to look at your position for a few different times. Calculate the following, rounding the component values to three decimal places. Draw the output vectors of $\vec{r}_8$ on Figure 10.3.1 in standard position.

i. $\vec{r}_8(3)$ ii. $\vec{r}_8(3.1)$ iii. $\vec{r}_8(3.14)$

The tips of these vectors correspond to the locations that would be sampled if you wanted to know the location of your car at $\vec{r}_8(\pi)$ but collected data every second, every tenth of a second, and every hundredth of a second, respectively.

(b) Write a couple of sentences to describe both geometrically and algebraically what happens to the output of $\vec{r}_8(t)$ as $t \to \pi$.

(c) Calculate $\vec{r}_8(\pi)$ and $\vec{r}_8(\pi) - \vec{r}_8(3)$. Sketch $\vec{r}_8(\pi)$ and $\vec{r}_8(3)$ in standard position in Figure 10.3.1. Also plot $\vec{r}_8(\pi) - \vec{r}_8(3)$, positioned in such a way to illustrate that it is difference of these two position vectors.

(d) Compute $\frac{\vec{r}_8(\pi)-\vec{r}_8(3)}{\pi-3}$ and explain how this calculation is different than the result of the previous step.

(e) How would you expect $\frac{\vec{r}_8(\pi)-\vec{r}_8(3.1)}{\pi-3.1}$ to be different than $\frac{\vec{r}_8(\pi)-\vec{r}_8(3)}{\pi-3}$? Use this idea write about what is measured by $\frac{\vec{r}_8(\pi)-\vec{r}_8(\pi-h)}{h}$ if we look at smaller and smaller values of h. Remember to be specific about what aspects of our curve or drive are being measured.

Activity 10.3.2. For each of the following vector-valued functions, state any values of a for which $\lim_{t \to a} \vec{r}(t)$ will *not* exist and find $\vec{r}\,'(t)$.

(a) $\vec{r}(t) = \langle \cos(t), t\sin(t), \ln(t) \rangle$.

(b) $\vec{r}(t) = \langle t^2 + 3t, e^{-2t}, \dfrac{t}{t^2+1} \rangle$.

(c) $\vec{r}(t) = \langle \tan(t), \cos(t^2), te^{-t} \rangle$.

(d) $\vec{r}(t) = \left\langle \sqrt{t^4+4}, \frac{2}{t^2+t}, e^{2t}\sin(-2t) \right\rangle$.

Activity 10.3.3.

(a) Let

$$\vec{r}(t) = \cos(t)\hat{\imath} - \sin(t)\hat{\jmath} + t\hat{k}.$$

Sketch the curve using some appropriate tool and make a drawing by hand that labels the point at the terminal point of $\vec{r}(\pi)$.

(b) Recall that we discussed earlier that the vector $\vec{r}\,'(a)$ is tangent to the graph of $\vec{r}(t)$ at the point where $t = a$. Find a direction vector for the line tangent to the graph of $\vec{r}$ at the point where $t = \pi$.

(c) Find the parametric equations of the line tangent to the graph of $\vec{r}$ when $t = \pi$.

(d) On your plot of the curve $\vec{r}(t)$, sketch the tangent line corresponding to $t = \pi$ and highlight the role of $\vec{r}\,'(\pi)$ on your plot.

Activity 10.3.4. Suppose a moving object in space has its velocity given by

$$\vec{v}(t) = (-2\sin(2t))\hat{\imath} + (2\cos(t))\hat{\jmath} + \left(1 - \frac{1}{1+t}\right)\hat{k}.$$

A graph of the position of the object for times t in $[-0.5, 3]$ is shown in Figure 10.3.9. Suppose further that the object is at the point $(1.5, -1, 0)$ at time $t = 0$.

(a) Determine $\vec{a}(t)$, the acceleration of the object at time t.

(b) Determine $\vec{r}(t)$, position of the object at time t.

(c) Compute the position, velocity, and acceleration vectors of the object at time $t = 1$ and plot these vectors using Figure 10.3.9.

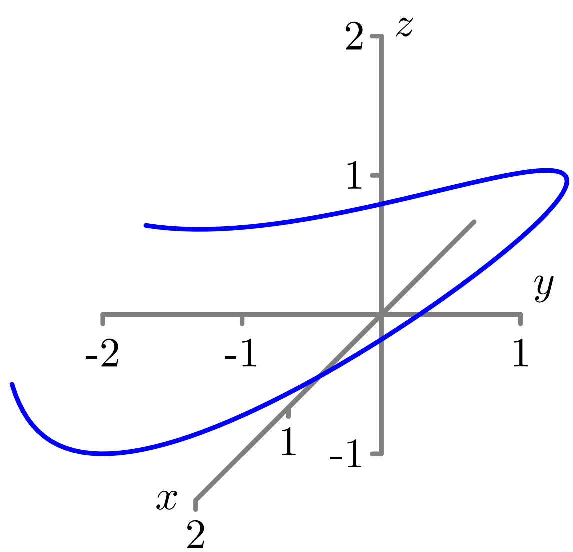


Figure 10.3.9 The position graph for the function in Activity 10.3.4

(d) Give the vector equation for the tangent line, $\vec{L}(t)$, that is tangent to the graph of $\vec{r}(t)$ at $t = 1$.

10.4 Arc Length

Preview Activity 10.4.1. In Preview Activity 10.2.1 and Preview Activity 10.3.1 we saw how the position reported by the location tracking system (LTS) of a self-driving car corresponds to a vector-valued function that describes the location of the car at a given time. We want to relate the location at different times to properties about how the car is being driven. In order to attract investors and drive shareholder value in *Steer Clear*, you decide to use the location tracking system's output to build a customized navigation and telemetry tool for a self-driving car. The first element you will need to build is a way to use the LTS output to calculate how far the car has driven in a given time.

(a) In your testing center (an abandoned parking garage), you drive your car around different levels of your testing center and note that the LTS recorded that the car had coordinates of $\langle 10, -5, 0\rangle$ at $t = 0$ and $\langle -8, 10, 12\rangle$ at $t = 30$ seconds. The units on each component of the coordinate vectors are meters. What is the distance between initial ($t = 0$) and final ($t = 30$) positions of the car? This distance is the length of the *displacement* vector of your total trip.

(b) Do you think the distance the car actually traveled is greater than, less than, or exactly the same as your answer to part (a)? Write a couple of sentences to explain your reasoning.

(c) In order to get more information for your telemetry system, you decide to pull the location data from your drive up the parking ramp every 10 seconds. The LTS gives the following data:

- $\langle 10, -5, 0\rangle$ at $t = 0$
- $\langle 6, 0, 6\rangle$ at $t = 10$
- $\langle -10, 2, 10\rangle$ at $t = 20$
- $\langle -8, 10, 12\rangle$ at $t = 30$

Calculate the distance between successive times and estimate how far your car went on the drive up the parking ramp.

(d) Do you think your estimate from part (c) is greater than, less than, or equal to the actual distance traveled? Write a couple of sentences to explain your reasoning.

(e) You look into the documentation on your location tracking system and see that you can specify how often the software will output location data. You decide that getting the location every half second seems like a good idea. How many data points will that correspond to for your drive up the ramp? Write a couple of sentences to describe what steps you would take with this new location data to calculate a better estimate on the distance traveled by your car. How might you go about finding the *exact* distance traveled by your car?

Activity 10.4.2. In this activity, we will use parameterizations to find the length of some common curves.

(a) Parameterize a circle of radius 3 centered at the origin. Give bounds on the parameter.

(b) Use your parameterization from the previous part in the definite integral of Theorem 10.4.5 to calculate the circumference of a circle of radius 3.

(c) Find the exact length of the spiral defined by $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), t\rangle$ on the interval $[0, 2\pi]$.

(d) Explain why your result for the length of the spiral is larger than the circumference of the circle of the same radius.

Notice here that while we focused on the circle of radius 3, changing to the more general radius of R for the circle gives a parameterization of $\langle R\cos(t), R\sin(t)\rangle$ with $0 \le t \le 2\pi$. The work to use the arc length integral to find the circumference of this circle requires only small modifications to the work you did in this activity.

Activity 10.4.3 Is it a property of the driver or the road?. In this activity, we want to determine if the different measurements we have described are a property of the driver or the road. A measurement is a **property of the driver** provided that the values of that measurement *can be different* for different drivers when measured at the same location on the racetrack. A measurement is a **property of the road** provided that different drivers *must have the same values* when measured at the same location on the racetrack.

(a) Let's start by looking at a couple of easy measurements. Is the time elapsed a property of the driver or the road? Be sure to explain your answer.

(b) Is position (the location of the car on the racetrack) a property of the driver or the road? Be sure to explain your answer.

(c) *Speedometer Reading.*

Now that we are warmed up, let's look at some more interesting measurements. The car's speedometer reading measures how fast (as a scalar) the car is moving. Is the car's speedometer reading a property of the driver or the road? Be sure to explain your answer.

(d) What vector calculus quantity is the speedometer reading?

(e) *Odometer Reading.*

The racecar's odometer measures the distance traveled by the car. Every car's odometer is set to be zero at the start of the race. Is the car's odometer reading a property of the driver or the road? Be sure to explain your answer.

10.5 The TNB Frame

Preview Activity 10.5.1. As CEO and Head of Engineering at *Steer Clear*, you decide that you are almost ready to start testing your self-driving car out on the road. Your navigation and telemetry software will use the location tracking system (LTS) to determine all of the important information about how the car is moving and how adjustments need to be made. Before you start programming your software, you decide to drive on a quiet, country road and collect data from the LTS to use as test data for your programming. In other words, you will drive on a section of road you already have mapped out in order to check that your software is calculating the correct information. The map in Figure 10.5.2 below shows the path you plan to take on the country road (with the direction given by the arrows on the plot). Notice that at points B and F, the road crosses itself to go in a different direction.

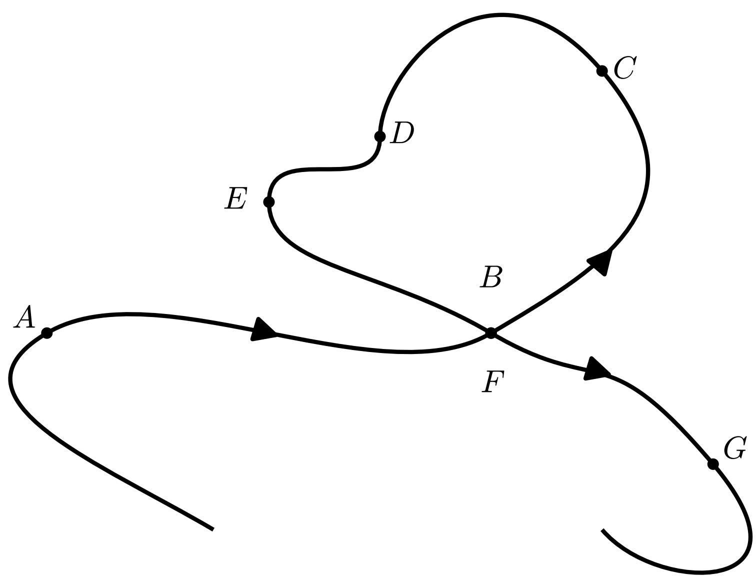


Figure 10.5.2 The path taken on a country road to collect data with your LTS

(a) At each of the labeled points on the curve, draw a vector in the direction of travel. Write a sentence about how you are determining this vector at the various points.

(b) At each point, decide whether the car is turning left or right. Write a sentence about how you are determining this turning based on the plot given.

Activity 10.5.2.

(a) Find $\vec{v}$, speed, and $\vec{T}$ as functions of t for the line parameterized as $\vec{r}(t) = \langle 3t-1, 2-2t, 5+t\rangle$. Write a few sentences about why your results make sense and why in for this particular curve $\vec{T}$ does not vary based on the parameter value.

(b) Find $\vec{v}$, speed, and $\vec{T}$ for the curve with parameterization $\vec{r}(t) = \langle t, t^2, t^3\rangle$.

(c) Consider a curve for which we do not know the parameterization. However, we do know that at a point P, the tangent line to the curve through P can be parameterized as $\langle 2-7t, 3t+1, -4t-1\rangle$. What are you able to say about $\vec{T}$ for this curve at the point P?

Activity 10.5.3.

(a) Draw an example of a curve such that $\vec{T}$ does not exist at a point on your curve. Identify the point where $\vec{T}$ fails to exist. Explain both why the direction of travel does not exist at that point based on your plot and why the direction of turning will not exist at the same point.

(b) Draw an example of a curve such that $\vec{T}\,' = \vec{0}$ at a point on your curve. Explain why $\vec{T}\,' = \vec{0}$ based on your plot and explain why $\vec{N}$ does not exist at the same point.

Activity 10.5.4. In general, calculating $\vec{N}$ as a function of the parameter t ends up being very difficult because there are multiple compositions of functions involved which means there is a nesting of chain rules involved in the definition. In this activity, we will examine how to go through the direct calculation of $\vec{N}$ for a curve parameterized by $\vec{r}(t) = \langle t, t^2, t^3 \rangle$.

(a) Calculate velocity $\vec{v}(t)$ and speed(t) for the parameterization $\vec{r}(t) = \langle t, t^2, t^3 \rangle$.

(b) Now calculate $\vec{T}(t)$ for the path given by $\vec{r}(t) = \langle t, t^2, t^3 \rangle$.

(c) Calculate the first component of $\vec{T}\,' = \frac{d\vec{T}}{dt}$.

(d) Calculate the second component of $\vec{T}\,'$.

(e) Calculate the third component of $\vec{T}\,'$.

(f) Put together your work for the three components of $\vec{T}\,'$.

(g) In the previous part, you likely decided that trying to simplify your formulas for $\frac{d\vec{T}}{dt}$ was more complicated than you had space for on your paper, and note that we still would need to calculate $\left\|\frac{d\vec{T}}{dt}\right\|$. Instead of doing more intricate algebraic computations, describe how you would calculate $\vec{N}$ if you had nice formulas for $\frac{d\vec{T}}{dt}$ and $\left\|\frac{d\vec{T}}{dt}\right\|$.

Activity 10.5.5.

(a) *Direction of Travel.*

Each race car has a 1 meter arrow attached to the hood of the car which points straight ahead. This arrow will measure the three dimensional vector that is the direction of travel. Is the direction of travel a property of the driver or the road? Be sure to explain your answer.

(b) *Turn Signal.*

The turn signal is measured by checking to see if the steering wheel is turned left or right. We are not measuring how much the steering wheel is turned in either direction, just whether the wheel is turned left or right. If the steering wheel is not turned either way at the instant we measure , then we say there is no turn signal because the car isn't turning at that instant. Is the turn signal a property of the driver or the road? Be sure to explain your answer.

(c) *Direction of Turning.*

The direction of turning is measured by using a 1 meter arrow that sticks perpendicularly out of the left side of the car when the steering wheel is turned left and out of the right side of the car when the steering wheel is turned right. We are not measuring how much the wheel is turned in either direction, just whether the wheel is turned left or right. If the steering wheel is not turned either way at the instant we measure the direction of turning, then we say there is no direction of turning because the car isn't turning at that instant. Is the direction of turning as a vector in $\mathbb{R}^3$ a property of the driver or the road? Be sure to explain your answer.

(d) Is $\vec{B}$ a property of the driver or the road? Justify your ideas.

10.6 Curvature

Preview Activity 10.6.1. As Chief Engineer and CEO at *Steer Clear*, you have done great work in transforming information from your location tracking system (LTS) into information on the position, velocity, and distance traveled by your car. In order to make the "self-driving" part of the self-driving car, you will need to compare information from the LTS (which describes how your car is moving) to GPS location data for the network of roads (which describes the paths your car should be taking). Since everyone on the road drives differently, you realize that you need to measure how quickly a stretch of road is turning in different places, which leads you to contact a friend who is a civil engineer for help understanding highway construction specifications.

(a) Figure 10.6.2 shows a map of a section of road on your testing route for your self-driving car with points labeled P_0 and P_1 which are 10 meters apart (along the road). Draw a vector in the direction of travel at both P_0 and P_1. Use these vectors to describe how the direction of travel is changing along the path from P_0 to P_1.

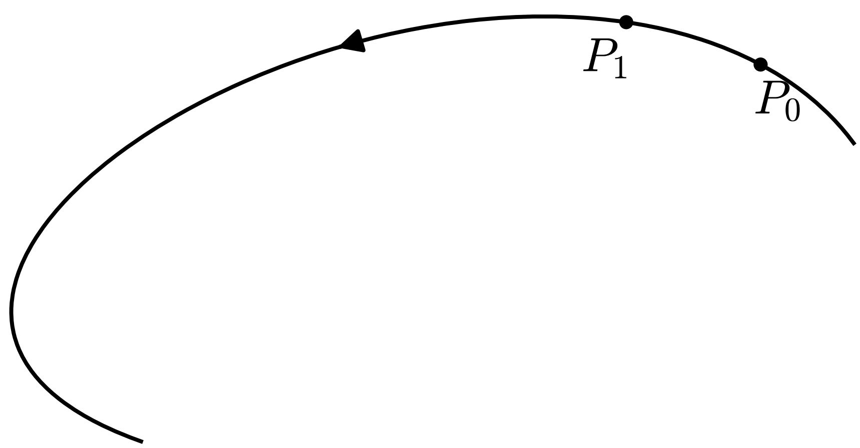


Figure 10.6.2 A plot of the section of road with points P_0 and P_1 separated by 10 meters

(b) Figure 10.6.3 shows a map of a section of road on your testing route for your self-driving car with points labeled Q_0 and Q_1 which are 10 meters apart (along the road). Draw a vector in the direction of travel at both Q_0 and Q_1. Use these vectors to describe how the direction of travel is changing along the path from Q_0 to Q_1.

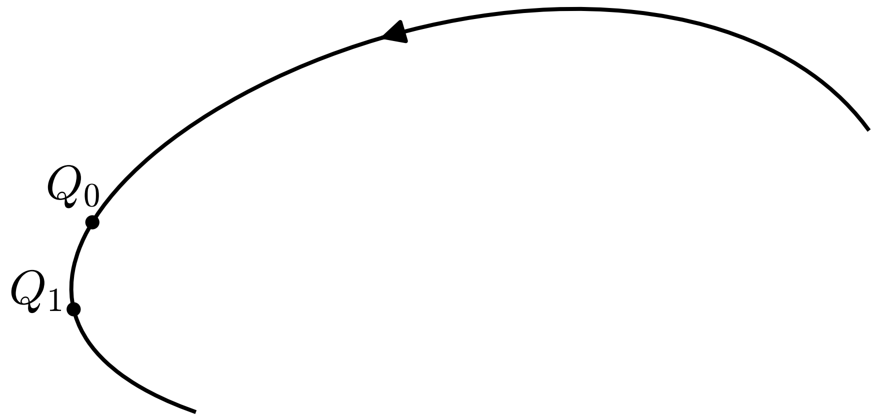


Figure 10.6.3 A plot of the section of road with points Q_0 and Q_1 separated by 10 meters

(c) Is the direction of travel changing faster over the path from P_0 to P_1 or over the path from Q_0 to Q_1. Write a few sentences to explain your reasoning and connect to your arguments and plots for the first two tasks.

Activity 10.6.2. In this activity we will use our previous work in finding unit speed parameterizations of lines and circles to make sense of the definition of curvature.

(a) Recall that in Example 10.4.8 we found that the unit speed parameterization of a line through the points $(1,2,3)$ and $(2,0,5)$ to be $\vec{r}_1(s) = \langle 1,2,3\rangle + \frac{s}{3}\langle 1,-2,2\rangle$. Calculate $\vec{T}(s)$, $\frac{d\vec{T}}{ds}$, and $\left\|\frac{d\vec{T}}{ds}\right\|$ for the unit speed parameterization of this line. Note that because we are using the unit speed parameterization, our parameter t is the same as the arc length traveled s. In other words, $t = s$ for this parameterization.

(b) Write a few sentences to explain why your result for the calculation of the curvature $\kappa = \left\|\frac{d\vec{T}}{ds}\right\|$ for this line makes sense. Exercise 15 asks you to do the calculations to confirm this property of an arbitrary line.

(c) Recall from Activity 10.4.2 have a parameterization of a circle in 2-space of radius R centered at the origin as $\langle R\cos(t), R\sin(t)\rangle$. A quick calculation will verify that this parameterization has constant speed and that speed is R.

Modifying this parameterization by multiplying the parameter by $\frac{1}{\text{speed}} = \frac{1}{R}$ gives the following parameterization:

$$\vec{r}(s) = \left\langle R\cos\left(\frac{s}{R}\right), R\sin\left(\frac{s}{R}\right)\right\rangle$$

Compute the speed of this parameterization to verify that this has unit speed. This will also show that $\vec{r}\,'(s) = \vec{v}(s) = \vec{T}(s)$. (Recall that this fact is not true in general!)

(d) Calculate $\vec{T}(s)$, $\frac{d\vec{T}}{ds}$, and $\left\|\frac{d\vec{T}}{ds}\right\|$ for the unit speed parameterization of a circle of radius R centered at the origin.

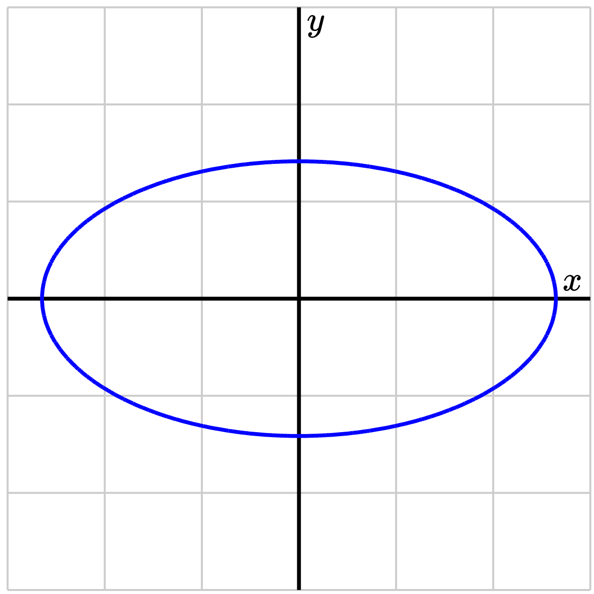
(e) Write a few sentences to explain why your result for the calculation of $\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$ of circle will be constant. You should also address why circles with larger radii will have have smaller curvature.

Activity 10.6.3. In this activity, we calculate the curvature of an ellipse and of a helix and use graphs to help with our understanding.

(a) Consider the ellipse shown below. We have omitted the scale on the axes because we want to think of this as the general ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which has parameterization

$$\vec{r}(t) = \langle a\cos(t), b\sin(t)\rangle.$$

(The ellipse pictured below has $a > b$.)



On the graph, identify the points at which you believe the curvature to be the largest and the points at which you believe the curvature is smallest.

(b) For the given parameterization of the ellipse, find $\vec{v}(t)$ and $\vec{a}(t)$. Also identify the parameter values that correspond to the points of the ellipse with the largest and smallest x or y coordinates.

(c) Use your calculations for $\vec{v}(t)$ and $\vec{a}(t)$ from part (b) in Equation (10.6.1) to find a formula for the curvature of the ellipse with the given parameterization. Use your formula to find the curvature of the ellipse at the parameter values you identified in part (b) as corresponding to the extreme points of the ellipse.

(d) While it is possible to use single-variable calculus techniques to find the parameter values that maximize and minimize $\kappa(t)$ on the interval $[0, 2\pi]$, the algebra involved masks much of the insight. Use graphing technology such as Desmos to plot the curvature function you found above, preferably with sliders that allow you to adjust the values of a and b. Write a couple of sentences about what you observe about the parameter values that maximize and minimize curvature and how they depend on a and b. Then use the general form of the curvature function from above to find the curvature at these parameter values, simplifying as much as possible.

(e) The standard helix has parameterization $\vec{r}_2(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath} + t\hat{k}$. Find the curvature of the helix using this parameterization.

(f) Write a few sentences to describe why the curvature of the helix given above is constant. You may want to include a plot of the curve.

Activity 10.6.4. In this activity we will look at the measurement of curvature and determine if curvature is a property of the driver or a property of the road. This is a continuation of Activity 10.4.3. As a reminder, a measurement is a property of the driver if the value(s) of that measurement *can* be different for different drivers (when measured at the same location on the racetrack). A measurement is a property of the road when different drivers *must* have the same value(s) (when measured at the same location on the racetrack).

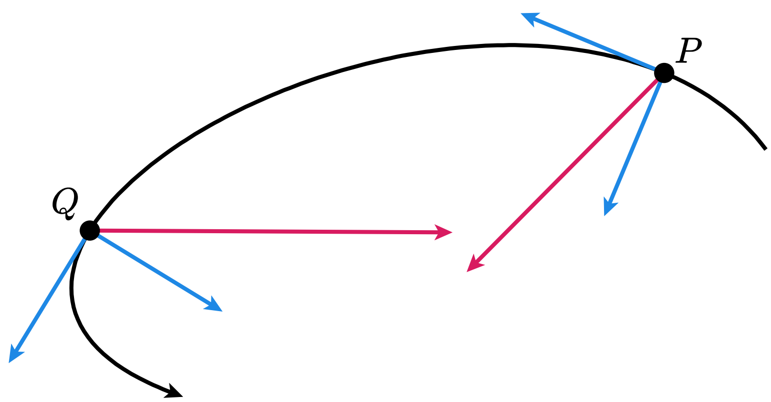
(a) Is curvature a property of the driver or the road? Write a few sentences about your reasoning and be sure to address why every driver may or may not have the same measurement at the same location on the track.

(b) What is the curvature on a straightaway (the race track is a line segment)? What would the radius of curvature be for a straightaway? Write a couple of sentences to justify your answers.

10.7 Splitting the Acceleration Vector

Preview Activity 10.7.1. All your good work as CEO and lead engineer at *Steer Clear* has started to pay off, literally. When you showed your work on using the location tracking system (LTS) to develop navigation and telemetry tools to an investment group (your grandparents), they were impressed and decided to give you money to further develop your self-driving car. You decided to use this infusion of cash to buy a gyroscopic accelerometer which will measure acceleration as a vector with magnitude and direction. In order to use your new instrument for information on the “driving” part of your self-driving car, you need to understand how your accelerometer readings will relate to the operations needed to drive a car.

(a) The image below shows the path you drove on a test drive along with points P and Q and the direction of travel. At the points P and Q, we have drawn three vectors. The acceleration vector $\vec{a}$ provided by your new gyroscopic accelerometer is shown in magenta. The two blue vectors are the unit tangent vector $\vec{T}$ and unit normal vector $\vec{N}$ at that point. Label each of the blue vectors as being $\vec{T}$ or $\vec{N}$.



(b) Draw the projection of $\vec{a}$ on $\vec{T}$ and the projection of $\vec{a}$ on $\vec{N}$ at point P and at point Q in the graph above.

(c) Is $\vec{a} \cdot \vec{T}$ positive, negative, or zero at P? What about at Q? Write a sentence or two to justify your answers.

(d) Is $\vec{a} \cdot \vec{N}$ is positive, negative, or zero at P? What about at Q? Write a sentence or two to justify your answers.

Activity 10.7.2. In this activity, we will work to understand the sign of $a_{\vec{T}}$ and $a_{\vec{N}}$.

(a) The tangential component of acceleration is defined as $a_{\vec{T}} = \vec{a} \cdot \vec{T}$, while equation (10.7.1) establishes that $a_{\vec{T}} = \frac{d(\text{speed})}{dt}$ is also true. Use these formulas to determine whether $a_{\vec{T}}$ can be zero. Either explain why $a_{\vec{T}}$ is always nonzero or describe scenarios in which $a_{\vec{T}} = 0$.

(b) Determine whether $a_{\vec{T}}$ can be negative. Either explain $a_{\vec{T}}$ is never negative or describe scenarios in which $a_{\vec{T}} < 0$

(c) The normal component of acceleration is defined as $a_{\vec{N}} = \vec{a} \cdot \vec{N}$, while equation (10.7.1) establishes that $a_{\vec{N}} = (\text{speed}) \left\| \frac{d\vec{T}}{dt} \right\|$ is also true. Use these formulas to determine whether $a_{\vec{N}}$ can be negative. Either explain why $a_{\vec{N}}$ is negative or describe scenarios in which $a_{\vec{N}} < 0$.

(d) Determine whether $a_{\vec{N}}$ can be zero. Either explain why $a_{\vec{N}}$ is always nonzero or describe scenarios in which $a_{\vec{N}} = 0$.

Activity 10.7.3. Compute the splitting of the acceleration for the curve given by $\vec{r}(t) = \langle t, \frac{t^2}{2}, \frac{t^3}{6} \rangle$ at $t = 1$. You should calculate $a_{\vec{T}}$, $a_{\vec{N}}$, $\vec{T}$, and $\vec{N}$. Additionally, you should verify $\vec{a} = a_{\vec{T}}\vec{T} + a_{\vec{N}}\vec{N}$ and that $\vec{T}$ is orthogonal to $\vec{N}$.

Activity 10.7.4 The Driver or the Road?. In this activity, you will determine if the measurements related to the splitting of acceleration are properties of the driver or properties of the road. This is a continuation of Activity 10.4.3. As a reminder, a measurement is a property of the driver if the value of that measurement ***can*** be different for different drivers when measured at the same location on the racetrack. A measurement is a property of the road when different drivers ***must*** have the same value(s) when measured at the same location on the racetrack.

(a) Pedal Usage: We measure the pedal usage as $+$, $-$, or 0 in the following way:

- $+$ if the gas pedal is being used
- $-$ if the brake pedal is being used
- 0 if no pedal is being used

Since our drivers are safety minded, they do not use more than one pedal at a time. Note that this measurement is a sign rather than a numerical value. Which vector calculus quantity's sign corresponds to the pedal usage? Write a sentence to explain your answer.

(b) Describe how the value of $a_{\vec{N}}$ would be felt by the driver in our analogy.

(c) In terms of the race car analogy, explain why $a_{\vec{N}}$ can't be negative.

(d) Write a few sentences about whether $a_{\vec{T}}$ is a property of the driver or a property of the road.

(e) Write a few sentences about whether $a_{\vec{N}}$ is a property of the driver or a property of the road.

11 Derivatives of Multivariable Functions

11.2 Functions of Several Variables

Preview Activity 11.2.1. Suppose you invest money in an account that pays 5% interest compounded continuously. If you have an initial investment of P dollars in the account, then A, the amount of money in the account after t years is given by

$$A = Pe^{0.05t}.$$

The variables P and t are independent of each other, so using functional notation we write

$$A(P,t) = Pe^{0.05t}.$$

(a) Find the amount of money in the account after 7 years if you originally invest 1000 dollars.

(b) Evaluate $A(5000, 8)$. Write a sentence to explain what this calculation represents.

(c) Now consider only the situation where the amount invested is fixed at 1000 dollars. Calculate the amount of money in the account after t years as indicated in the table below. Round payments to the nearest penny.

Duration (in years)	2	3	4	5	6
Amount (dollars)					

(d) Now consider the situation where we want to know the amount of money in the account after 10 years given various initial investments. Calculate the amount of money in the account as indicated in the table below. Round payments to the nearest penny.

Initial investment (dollars)	500	1000	5000	7500	10000
Amount (dollars)					

(e) Describe as best you can what combinations of initial investments and time will result in an account containing \$10,000.

Activity 11.2.2. Identify the domain and range of each of the following functions. Sketch each domain in the xy-plane.

(a) $f(x, y) = x^2 + y^2$

(b) $f(x, y) = \sqrt{x^2 + y^2}$

(c) $Q(x, y) = \dfrac{x + y}{x^2 - y^2}$

(d) $s(x, y) = \dfrac{1}{\sqrt{1 - xy^2}}$

Activity 11.2.3. Complete Table 11.2.6 by filling in the missing values of the function f. Round entries to the nearest tenth.

Table 11.2.6 Values of $f(x,y) = \dfrac{x^2 \sin(2y)}{g}$ **with** $g = 32\mathbf{ft/s}^2$

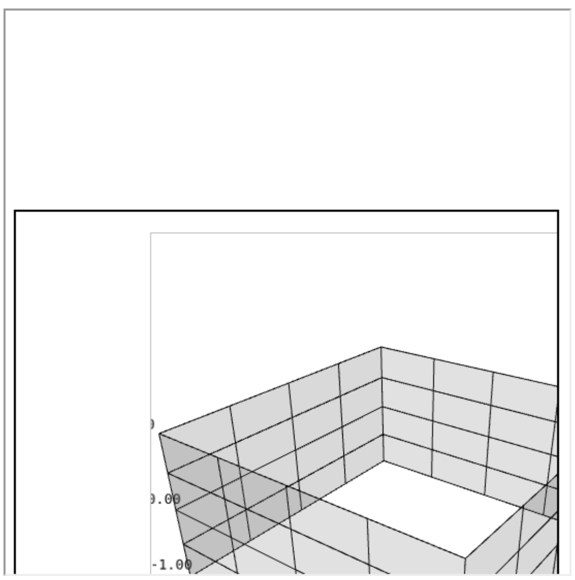
$x \Downarrow \backslash y \Rightarrow$	0.4	0.6	0.8	1.0	1.2
50	56.0	72.8	78.1	71.0	
75		163.8	175.7	159.8	118.7
100	224.2	291.3	312.4	284.2	211.1
125	350.3	455.1		444.0	329.8
150	504.4	655.3	702.8	639.3	474.9
175	686.5	892.0	956.6	870.2	

Activity 11.2.4. In this activity, we investigate the use of traces to better understand a function through both tables and graphs.

(a) Identify the $y = 0.6$ trace for the distance function f defined by $f(x, y) = \frac{x^2 \sin(2y)}{g}$ by highlighting or circling the appropriate cells in Table 11.2.6. Write a sentence to describe the behavior of the function along this trace.

(b) Identify the $x = 150$ trace for the distance function by highlighting or circling the appropriate cells in Table 11.2.6. Write a sentence to describe the behavior of the function along this trace.

(c) In the next several parts, we will be looking at using traces to help us draw an accurate plot of the surface given by $z = g(x, y) = yx^2$. As a first step, find the equation for the $y = 1$ trace of $z = g(x, y) = yx^2$ and draw a graph of the $y = 1$ trace on the corresponding face in Figure 11.2.13.



Standalone

In Context

Embed

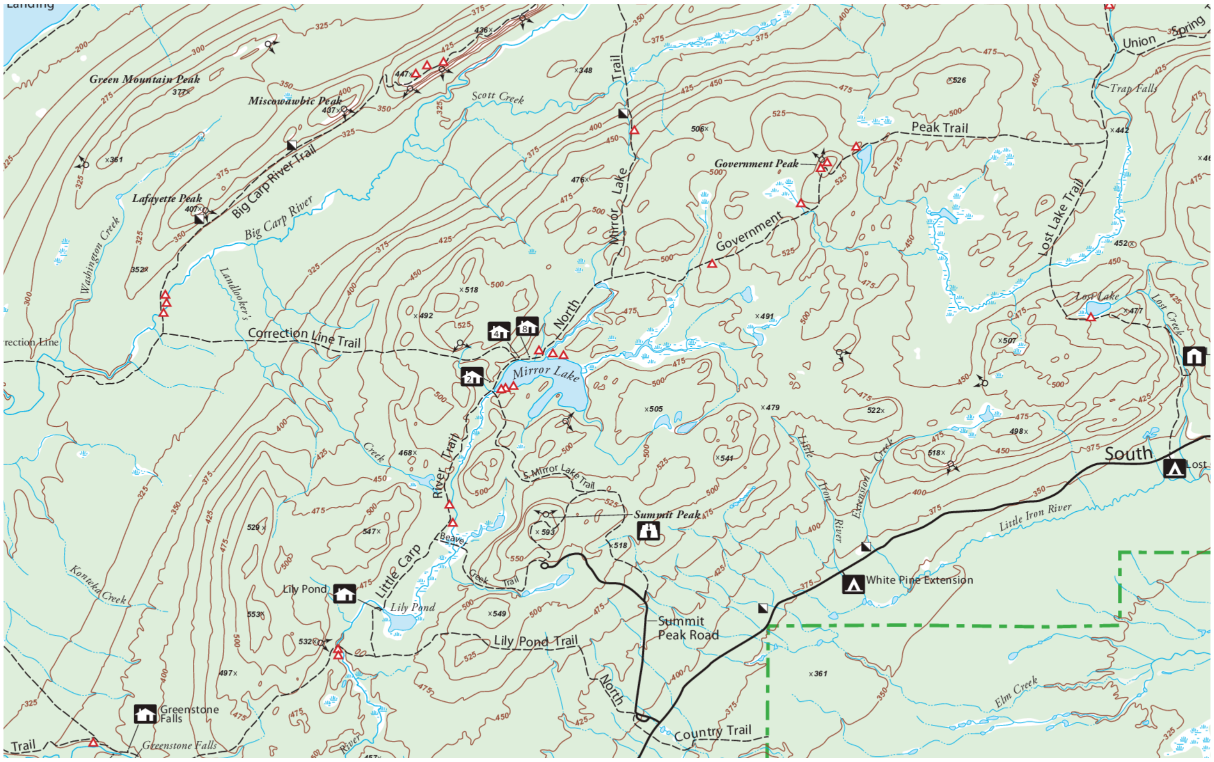
Figure 11.2.13 A bounding box for the region with $-1 \leq x, y, z \leq 1$

(d) Find the equation for each of the following traces and draw a plot of the trace on the corresponding face in Figure 11.2.13.

- $y = -1$
- $x = 1$
- $x = -1$

(e) Draw a few traces (at least three more) that correspond to values in the middle of the plot to fill in a plot of the surface given by $z = x^2y$.

Activity 11.2.5. Use the topographical map of the Porcupine Mountains below to answer the following questions. Note that points of interest are sometimes marked with an X and have their elevation listed.



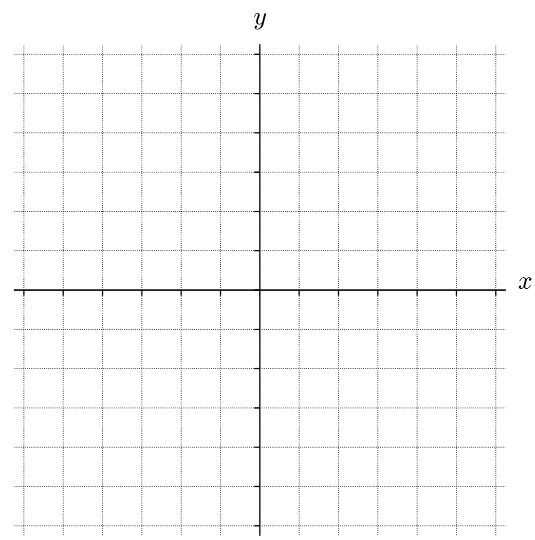
(a) Identify the highest and lowest points you can find.

(b) Describe how your elevation would change if you walked in a straight line from the lowest to the highest elevation points. You may want to sketch a plot of the elevation along your path.

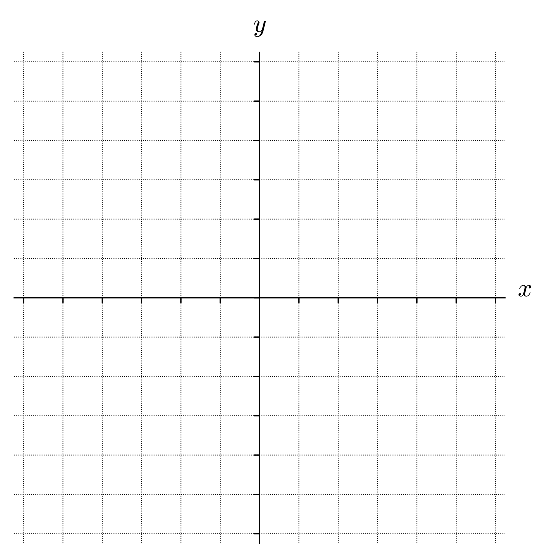
(c) If you walk along the Big Carp River Trail (in the top left part of the image) from the left to the right as shown on the map, describe which parts of the trail will have steep increases in elevation and which parts you think will be the most like level ground.

Activity 11.2.6. In this activity, you will make contour plots by hand and look at how the spacing of contours in your plot should give an idea about how different surface shapes can be distinguished.

(a) Let $f(x,y) = x^2 + y^2$. Draw the level curves $f(x,y) = k$ for $k = 1$, $k = 2$, $k = 3$, and $k = 4$ on the axes below. Be sure to label the scale of the axes. Explain what the surface defined by f looks like.



(b) Let $g(x,y) = \sqrt{x^2 + y^2}$. Draw the level curves $g(x,y) = k$ for $k = 1$, $k = 2$, $k = 3$, and $k = 4$ on the axes below. Use the same scale on these axes as in the previous part. Explain what the surface defined by g looks like.



(c) Compare and contrast the graphs of f and g. How are they alike? How are they different? Use traces for each function to help answer these questions.

11.3 Limits

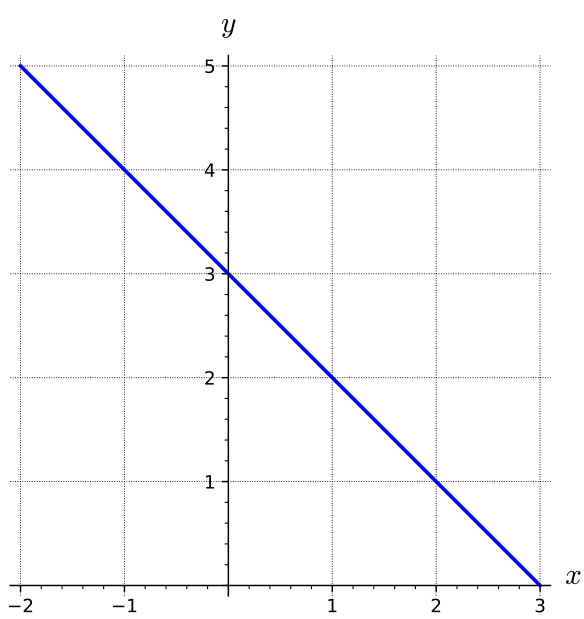
Preview Activity 11.3.1. In this Preview Activity, we will recall and work with several ideas related to limits of single-variable functions by working with tables, graphs, and the algebraic forms of these functions.

(a) Let f defined by $f(x) = 3 - x$. Complete the table below.

x	-0.2	-0.1	-0.01	0.01	0.1	0.2
$f(x)$						

(b) Write a sentence about what the table in part (a) suggests regarding $\lim_{x\to 0} f(x)$.

(c) Write a couple of sentences to explain how your values in the table in part (a) and $\lim_{x\to 0} f(x)$ are demonstrated in the graph below.



(d) Let $g(x) = \dfrac{x}{|x|}$. Fill in the blanks of the table below with the appropriate outputs of g for values near $x = 0$. Note that g is not defined at $x = 0$.

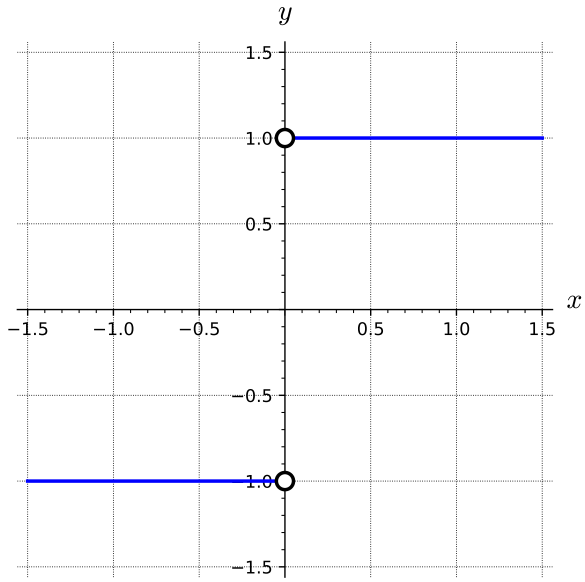
x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$g(x)$						

(e) Write a few sentences about what the values in the table in part (d) suggests about $\lim_{x\to 0} g(x)$.

(f) Write a couple of sentences to explain how your values in the table in part (d) and

$$\lim_{x\to 0} g(x)$$

are demonstrated in the plot below.



(g) Let $h(x) = \dfrac{1-x^2}{x+1}$. At what point(s) is h undefined?

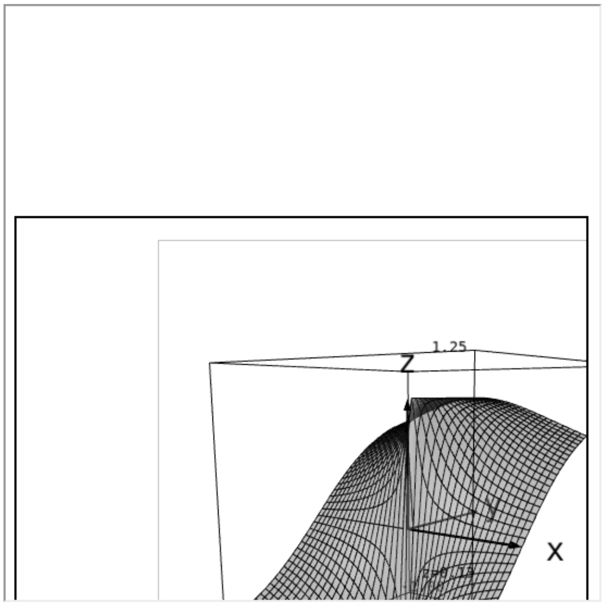
(h) Show that the limit as x goes to -1 of $h(x)$ exists. Write a sentence or two about why $h(-1)$ does not exist but the limit as x goes to -1 does exist.

(i) Graph $h(x)$ for a region around $x = -1$ and explain how this graph relates to your answer to the previous two parts.

Activity 11.3.2. Consider the function f, defined by

$$f(x,y) = \frac{y}{\sqrt{x^2+y^2}}.$$

The graph of this function is shown below.



Standalone

In Context

Embed

(a) Is f defined at the point $(0,0)$? What, if anything, does this say about whether f has a limit at the point $(0,0)$?

(b) Values of f (to three decimal places) at several points close to $(0,0)$ are shown below.

$x\backslash y$	-1.000	-0.100	0.000	0.100	1.000
-1.000	-0.707	—	0.000	—	0.707
-0.100	—	-0.707	0.000	0.707	—
0.000	-1.000	-1.000	—	1.000	1.000
0.100	—	-0.707	0.000	0.707	—
1.000	-0.707	—	0.000	—	0.707

Based on these calculations, state whether you think f has a limit at $(0,0)$ and give an argument supporting your statement.

(c) Now we formalize your conjecture from the previous part by considering what happens if we restrict our attention to different paths. First, we look at f for points in the domain along the x-axis. That is, we consider what happens when $y = 0$. Write out what the output of f will when $y = 0$. In other words, what is $f(x, 0)$ equal to?

What is the behavior of $f(x, 0)$ as $x \to 0$? If we approach $(0, 0)$ by moving along the x-axis, what value do we find as the limit?

(d) What is the behavior of f along the line $y = x$ when $x > 0$. That is, what is the value of $f(x, x)$ when $x > 0$? If we approach $(0, 0)$ by moving along the line $y = x$ in the first quadrant, what value do we find as the limit?

(e) In general, if $\lim_{(x,y)\to(0,0)} f(x, y) = L$, then $f(x, y)$ approaches L as (x, y) approaches $(0, 0)$, regardless of the path we take in letting $(x, y) \to (0, 0)$. Based on the last two parts of this activity, can the limit

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

exist? Write a sentence or two to explain your reasoning.

(f) Substitute $y = mx$ into $f(x, y)$ and simplify your result for $f(x, mx)$. Use your expression and limit rules to evaluate

$$\lim_{x\to 0} f(x, mx).$$

Use this limit to explain the the behavior of $f(x, y)$ as (x, y) approaches $(0, 0)$ along lines of the form $y = mx$ and how this observation reinforces your conclusion about the existence of $\lim_{(x,y)\to(0,0)} f(x, y)$ from the previous part of this activity.

Activity 11.3.3. Define the function g by

$$g(x, y) = \frac{x^2 y}{x^4 + y^2}.$$

We will investigate the limit $\lim_{(x,y)\to(0,0)} g(x, y)$.

(a) What is the behavior of g on the x-axis? That is, what is $g(x, 0)$ and what is the limit of g as (x, y) approaches $(0, 0)$ along the x-axis?

(b) What is the behavior of g on the y-axis? That is, what is $g(0, y)$ and what is the limit of g as (x, y) approaches $(0, 0)$ along the y-axis?

(c) What is the behavior of g on the line $y = mx$? That is, what is $g(x, mx)$ and what is the limit of g as (x, y) approaches $(0, 0)$ along the line $y = mx$?

(d) Based on what you have seen so far, do you think $\lim_{(x,y)\to(0,0)} g(x, y)$ exists? If so, what do you think its value is?

(e) Now consider the behavior of g on the parabola $y = x^2$? What is $g(x, x^2)$ and what is the limit of g as (x, y) approaches $(0, 0)$ along this parabola?

(f) State whether you think the limit $\lim_{(x,y)\to(0,0)} g(x, y)$ exists or not and provide a justification of your statement.

Activity 11.3.4. In this activity, we again consider the limit as (x, y) goes to $(0, 0)$ of $g(x, y) = \frac{x^2y^2}{x^2+y^2}$, which we previously saw in Example 11.3.9.

(a) Convert g to polar coordinates.

(b) Find the limit as $r \to 0$ of your expression in polar coordinates.

(c) Write a couple of sentences explaining how your answer to the previous part shows that the original limit is 0 and contrast with the previous example.

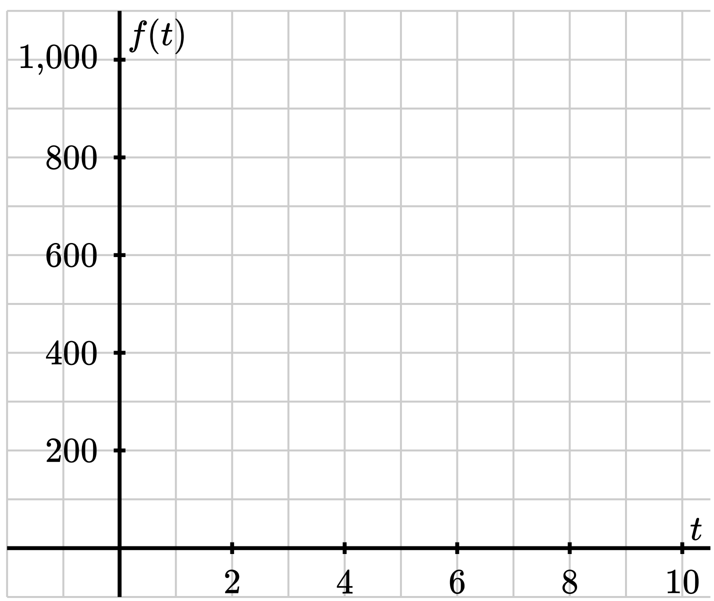
11.4 First-Order Partial Derivatives

Preview Activity 11.4.1. Suppose we take out an \$18,000 car loan at interest rate r and we agree to pay off the loan in t years. The monthly payment, in dollars, is

$$M(r,t) = \frac{1500r}{1 - \left(1 + \frac{r}{12}\right)^{-12t}}.$$

(a) If you take a loan with an interest rate of 3% (so $r = 0.03$) and will pay it off in $t = 4$ years, what will your monthly payment be?

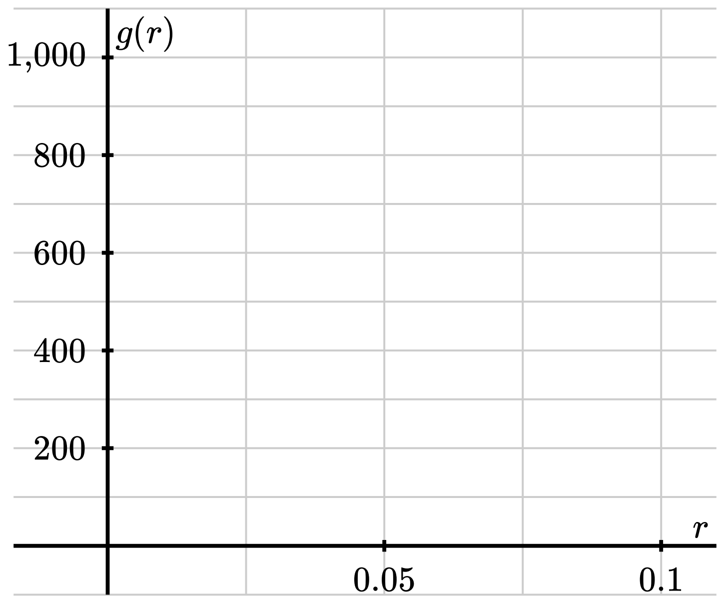
(b) Suppose the interest rate on loans is fixed at 3%. Express M as a function f of t alone using $r = 0.03$. That is, let $f(t) = M(0.03, t)$. Sketch the graph of f on the axes below and write a couple of sentences to explain the meaning of the function f.



(c) Find the instantaneous rate of change $f'(4)$ and state the units on this quantity. What information does

$f'(4)$ tell us about our car loan? What information does $f'(4)$ tell us about the graph you sketched in (b)?

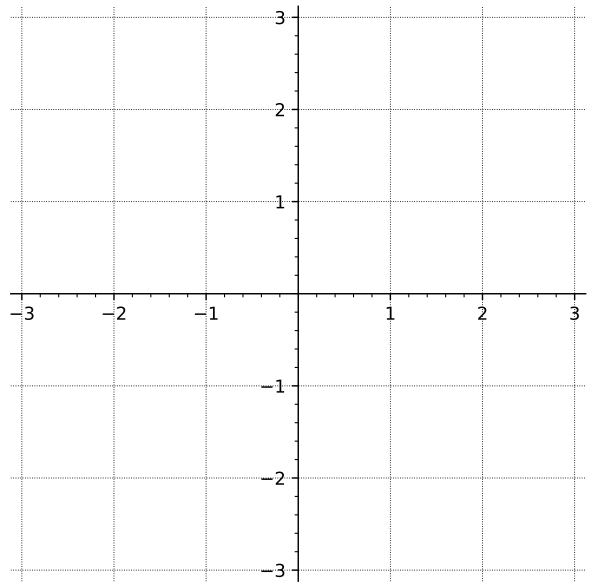
(d) Suppose that we want to examine how the monthly payment will change if we plan to take 4 years to pay off the loan. Express M as a function of the interest rate r with a fixed loan term of $t = 4$. That is, let $g(r) = M(r, 4)$. Give the function $g(r)$ and sketch a graph of g on the axes provided. Write a couple of sentences to explain the meaning of the function g.



(e) Find the instantaneous rate of change $g'(0.03)$ and state the units on this quantity. Write a couple of sentences to explain what $g'(0.03)$ tell us about the car loan. What information does $g'(0.03)$ tell us about the graph you sketched in (d)?

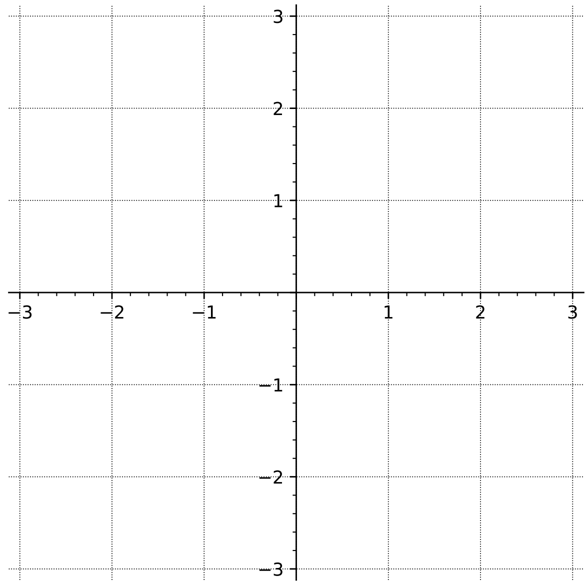
Activity 11.4.2. In this activity, we will be examining the function $f(x,y) = \dfrac{xy^2}{x+1}$ at the point $(1,2)$. Specifically, we will be looking at how the traces of this function will be related to the values of the partial derivatives.

(a) Write a formula for the trace $f(x,2)$ as a function of x. On the axes provided, draw the graph of the trace with $y=2$ around the point where $x=1$. Then sketch the tangent line at the point with $x=1$ on your graph.



(b) Compute the partial derivative $f_x(1,2)$ and relate its value to the sketch you just made.

(c) Write a formula for the trace $f(1,y)$ as a function of y. On the axes provided, draw the graph of the trace with $x=1$. Then sketch the tangent line at the point with $y=2$ on your graph.



(d) Compute the partial derivative $f_y(1, 2)$ and relate its value to the sketch you just made.

Activity 11.4.3.

(a) If $f(x, y) = 3x^3 - 2x^2y^5$, find the partial derivatives f_x and f_y.

(b) If $f(x, y) = \dfrac{xy^2}{x+1}$, compute $\partial f/\partial x$ and $\partial f/\partial y$.

(c) If $g(r, s) = rs\cos(r)$, find the partial derivatives g_r and g_s.

(d) Find all possible first-order partial derivatives of $q(x, t, z) = \dfrac{x2^tz^3}{1+x^2}$.

Activity 11.4.4. The speed of sound C traveling through ocean water is a function of temperature, salinity and depth. It may be modeled by the function

$$\begin{aligned} C = &\ 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 \\ &+ (1.34 - 0.01T)(S - 35) + 0.016D. \end{aligned}$$

Here C is the speed of sound in meters/second, T is the temperature in degrees Celsius, S is the salinity in grams/liter of water, and D is the depth below the ocean surface in meters.

(a) State the units for each of the partial derivatives, C_T, C_S and C_D. For each partial derivative, write a sentence that explains the physical meaning of the partial derivatives in terms of the rate of change for C.

(b) Use algebraic rules to find the partial derivatives C_T, C_S and C_D.

(c) Evaluate each of the three partial derivatives at the point where $T = 10$, $S = 35$ and $D = 100$. Write a few sentences to explain what the sign of each partial derivatives tell us about the behavior of the function C at the point $(10, 35, 100)$.

Activity 11.4.5. The wind chill, as frequently reported, is a measure of how cold it feels outside when the wind is blowing. In the table below, the wind chill w is a function of the wind speed v and the ambient air temperature T. Both w and T are measured in degrees Fahrenheit, while v is measured in miles per hour. This means that w can be expressed in the form $w = w(v, T)$.

$v \backslash T$	-20	-15	-10	-5	0	5	10
10	-41	-35	-28	-22	-16	-10	-4
15	-45	-39	-32	-26	-19	-13	-7
20	-48	-42	-35	-29	-22	-15	-9
25	-51	-44	-37	-31	-24	-17	-11
30	-53	-46	-39	-33	-26	-19	-12
35	-55	-48	-41	-34	-27	-21	-14

(a) When computing the partial derivative with respect to wind speed w_v, which variable is held constant and which variable changes? Do you think w_v is positive, negative, or zero at $(v, T) = (20, -10)$? Write a couple of sentences to explain your reasoning.

(b) When computing the partial derivative with respect to temperature w_T, which variable is held constant and which variable changes? Do you think w_T is positive, negative, or zero at $(v, T) = (20, -10)$? Write a couple of sentences to explain your reasoning.

(c) Recall that we can estimate a partial derivative of a single variable function f using the central difference $\frac{f(x+h)-f(x-h)}{2h}$ for small values of h. A partial derivative is a derivative of an appropriate trace, so by considering a difference quotient for a trace of w, estimate the partial derivative $w_v(20, -10)$. Include units on your answer and write a sentence explaining the meaning of the value of $w_v(20, -10)$.

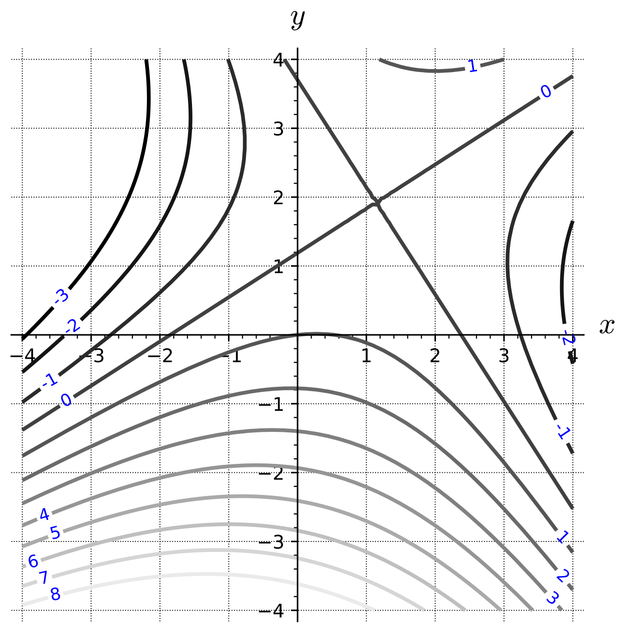
(d) Recall from single-variable calculus that for a function f, $f(a + h) \approx f(a) + hf'(a)$. Use this idea and your result above to estimate the wind chill $w(18, -10)$.

(e) Estimate the partial derivative $w_T(20, -10)$. Include units on your answer and write a sentence explaining the meaning of the value of $w_T(20, -10)$.

(f) Use your result above to estimate the wind chill $w(20, -12)$.

(g) Consider how you might combine your previous results to estimate the wind chill $w(18, -12)$. Explain your process.

Activity 11.4.6. Shown below is a contour plot of a function f. The output values of the function are indicated on the contours. This activity will ask you to use this contour plot to estimate some partial derivatives of f.



(a) We first wish to estimate the partial derivative $f_x(-2,-1)$.

(i) When computing f_x, which variable is treated as a constant? On the contour plot, sketch a line segment for which that variable is constant through the point $(-2,-1)$.

(ii) To estimate $f_x(-2,-1)$ using a difference quotient, you must identify two points in the xy-plane that are near $(-2,-1)$, on the line segment you sketched in the previous part (or an extension of that line segment), and for which you can use the contour plot to find the value of the function at those points. Mark these points, which we will call (a_1,b_1) and (a_2,b_2) on your contour plot and record the values of f at these points.

(iii) Use a difference quotient to estimate $f_x(-2,-1)$.

(b) In this part and the next, you will need to follow a similar process to the previous part. However, the activity does not break down the steps each time. Estimate the partial derivative $f_y(-2,-1)$.

(c) Estimate the partial derivatives $f_x(1,-1)$ and $f_y(1,-1)$.

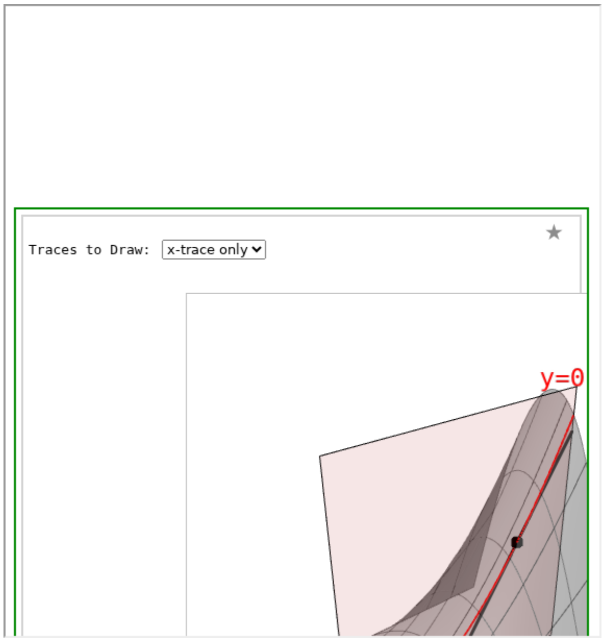
(d) Locate, if possible, one point (x,y) where $f_x(x,y)=0$.

(e) Locate, if possible, one point (x,y) where $f_x(x,y)<0$.

(f) Locate, if possible, one point (x,y) where $f_y(x,y)>0$.

11.5 Second-Order Partial Derivatives

Preview Activity 11.5.1. Once again, we consider the function f defined by $f(x, y) = \frac{x^2 \sin(2y)}{32}$ that measures a projectile's range as a function of its initial speed x and launch angle y. The graph of this function, including traces with $x = 150$ and $y = 0.6$, is shown in the interactive graph below.



Standalone

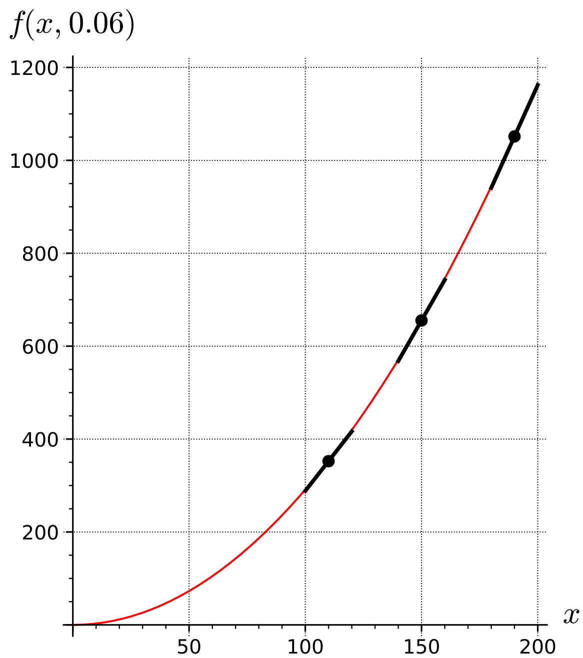
In Context

Embed

(a) Compute the partial derivatives f_x and f_y as functions of x and y.

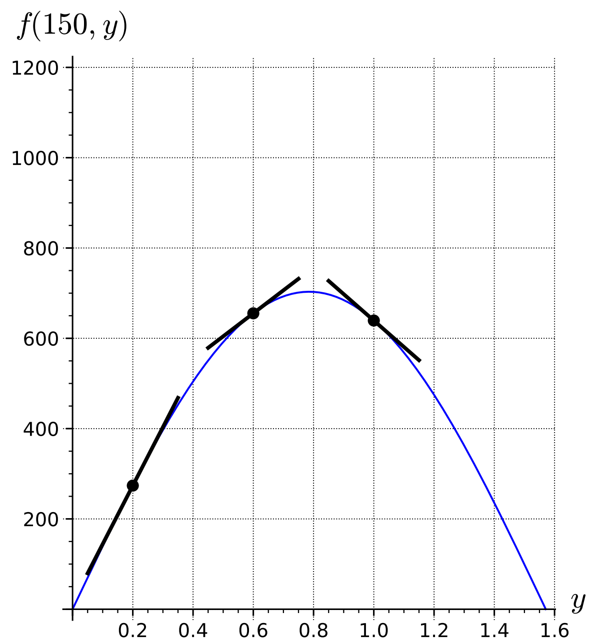
(b) Notice that f_x itself is a new function of x and y, so we may now compute the partial derivatives of f_x. Find the partial derivative $\frac{\partial}{\partial x}\left[f_x\right]$, which we will denote by f_{xx}. Verify that $f_{xx}(150, 0.6) \approx 0.058$.

(c) The graph below shows the trace of f with $y = 0.6$ with three tangent lines included. Write a few sentences to explain how $f_{xx}(150, 0.6) \approx 0.058$ is reflected in this figure.



(d) Find the partial derivative $\frac{\partial}{\partial y}\left[f_y\right]$, which we will denote by f_{yy} and compute the value of $f_{yy}(150, 0.6)$.

(e) The graph below shows the trace $f(150, y)$ and includes three tangent lines. Write a couple of sentences to explain how the value of $f_{yy}(150, 0.6)$ is reflected in this figure.



(f) Because f_x and f_y are each functions of *both* x and y, they each have *two* partial derivatives. You have already computed two of them. Now compute

$$\frac{\partial}{\partial y}\left[f_x\right] \qquad \text{and} \qquad \frac{\partial}{\partial x}\left[f_y\right]$$

for the range function $f(x, y) = \frac{x^2 \sin(2y)}{32}$ by using your earlier computations of f_x and f_y. Write a sentence to explain how you calculated these “mixed” partial derivatives.

Activity 11.5.2.

(a) Find all second-order partial derivatives of the following functions. For each partial derivative you calculate, state explicitly which variable is being held constant.

(i) $f(x, y) = x^2y^3$

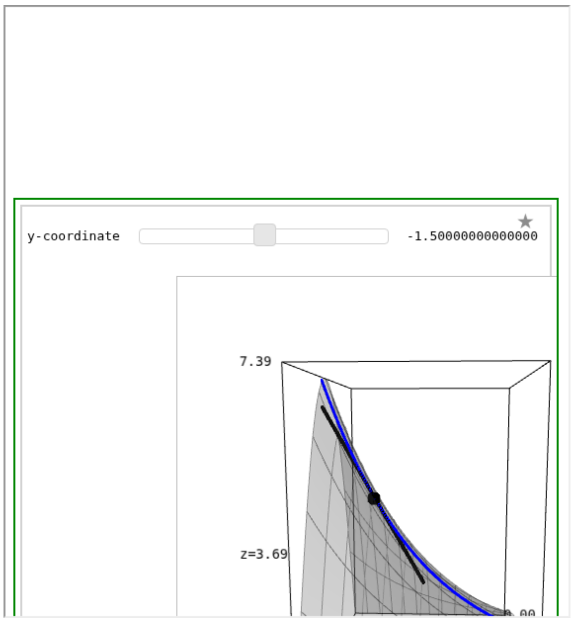
(ii) $y(q, p) = p\cos(q)$

(iii) $g(s, t) = st^3 + s^4$

(b) If $h(x, y, z, t) = 9x^9z - xyz^9 + 9t$, how many second-order partial derivatives does the function h have? Write a sentence to justify your reasoning on the number of second-order partial derivatives of h. Finally, find h_{xz} and h_{zx} (you do not need to find the other second-order partial derivatives).

Activity 11.5.3. We will continue to work with the function f defined by $f(x,y) = \sin(x)e^{-y}$.

(a) In the interactive graphic below we see the trace of $f(x,y) = \sin(x)e^{-y}$ with x held constant with $x = 1.75$ plotted in blue. Use the slider to investigate how the slope of the tangent line changes as you vary the y-coordinate along this trace. Write a couple of sentences that describe whether the slope of the tangent lines to this curve increase or decrease as y increases along the trace $x = 1.75$. Be sure to pay attention to which direction corresponds to each coordinate increasing.



Standalone

In Context

Embed

(b) Compute $f_{yy}(x,y)$ algebraically and explain how your observations in the previous part are related to the value of $f_{yy}(1.75,y)$. Your response should address the notion of concavity. Be careful to note the directions in which y is increasing.

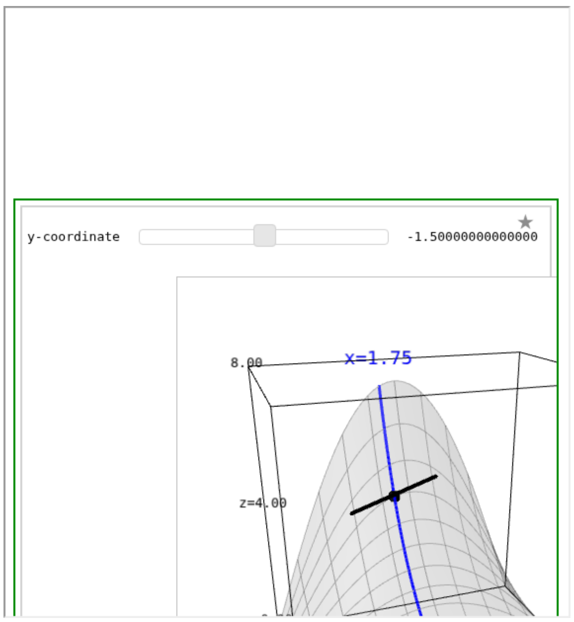
(c) We want to explore what the mixed partial derivative f_{xy} describes. We will do this by focusing on the point $(x,y) = (1.75,-1.5)$. In this part, we will work through the definition for $f_{xy}(1.75,-1.5)$ carefully.

The partial derivative $f_x(1.75,-1.5)$ measures the slope of the line tangent to the trace given by $y = -1.5$. These tangent lines change in the x-direction (parallel to the x-axis). When we consider f_{xy}, we are taking the partial derivative of f_x with respect to y, $\frac{\partial}{\partial y}\left[f_x\right]$. Thus, we are considering how the slope of the tangent line in the x-direction changes when we vary the y-coordinate a small amount. We can approximate $f_{xy}(1.75, -1.5)$ with the difference quotient

$$\frac{f_x(1.75, -1.5 + h) - f_x(1.75, -1.5)}{h}.$$

One way of visualizing this is by thinking of sliding a pencil along the trace with $x = 1.5$ so that the pencil is in the x-direction and tangent to the surface. In the interactive image below, the pencil would be the black tangent line drawn. Use the slider at the top of the interactive to change the y-coordinate of the point where the tangent line is drawn. Examine what happens to the slope of the tangent line as you increase the y-coordinate of the point of tangency by adjusting the slider. The tangent line in the x-direction at the point $(1.75, -1.5)$ is drawn in gray for reference purposes.

Based on your exploration, write a few sentences about whether $f_{xy}(1.75, -1.5)$ is positive or negative and justify your reasoning.



Standalone

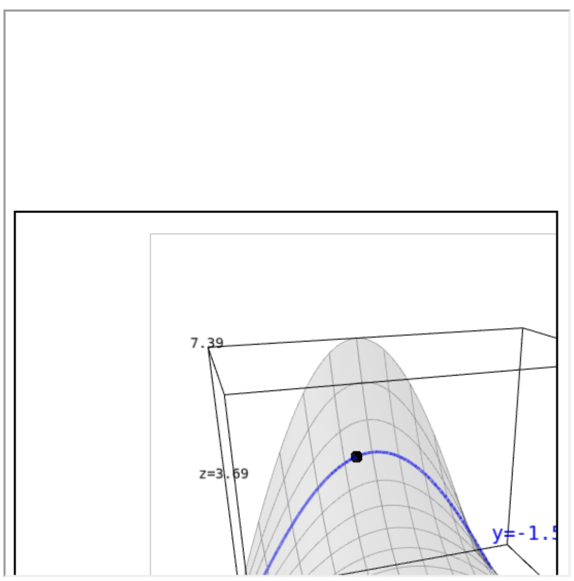
In Context

Embed

(d) Compute $f_{xy}(x, y)$ algebraically and evaluate $f_{xy}(1.75, -1.5)$. Write a couple of sentences about how this value compares with your observations in the previous part.

(e) We know that $f_{xx}(1.75, -1.5)$ measures the concavity of the $y = -1.5$ trace, and that $f_{yy}(1.75, -1.5)$ measures the concavity of the $x = 1.75$ trace. What do you think the quantity $f_{xy}(1.75, -1.5)$ measures?

(f) In the interactive image below, the trace with $y = -1.5$ is highlighted with the point $(1.75, -1.5, f(1.75, -1.5))$ drawn in black. Sketch three tangent lines to this trace whose slopes correspond to the value of $f_{yx}(x, -1.5)$ for three different values of x near $x = 1.75$. Use your tangent lines to state whether $f_{yx}(1.75, -1.5)$ is positive or negative. Justify your reasoning and describe what you think $f_{yx}(1.75, -1.5)$ measures.



Standalone

In Context

Embed

Activity 11.5.4. As we saw in Activity 11.4.5, the wind chill $w(v, T)$, in degrees Fahrenheit, is a function of the wind speed, in miles per hour, and the air temperature, in degrees Fahrenheit. Some values of the wind chill are recorded in the table below.

$v\backslash T$	-20	-15	-10	-5	0	5	10
10	-41	-35	-28	-22	-16	-10	-4
15	-45	-39	-32	-26	-19	-13	-7
20	-48	-42	-35	-29	-22	-15	-9
25	-51	-44	-37	-31	-24	-17	-11
30	-53	-46	-39	-33	-26	-19	-12
35	-55	-48	-41	-34	-27	-21	-14

(a) Estimate the partial derivatives $w_T(20, -15)$, $w_T(20, -10)$, and $w_T(20, -5)$. Use these results to estimate the second-order partial $w_{TT}(20, -10)$.

(b) In a similar way, estimate the second-order partial $w_{vv}(20, -10)$.

(c) Estimate the partial derivatives $w_T(20, -10)$, $w_T(25, -10)$, and $w_T(15, -10)$, and use your results to estimate the partial $w_{Tv}(20, -10)$.

(d) In a similar way, estimate the partial derivative $w_{vT}(20, -10)$.

(e) Write a few sentences to explain what the values $w_{TT}(20,-10)$, $w_{vv}(20,-10)$, and $w_{Tv}(20,-10)$ indicate regarding the behavior of $w(v,T)$.

11.6 Linearization: Tangent Planes and Differentials

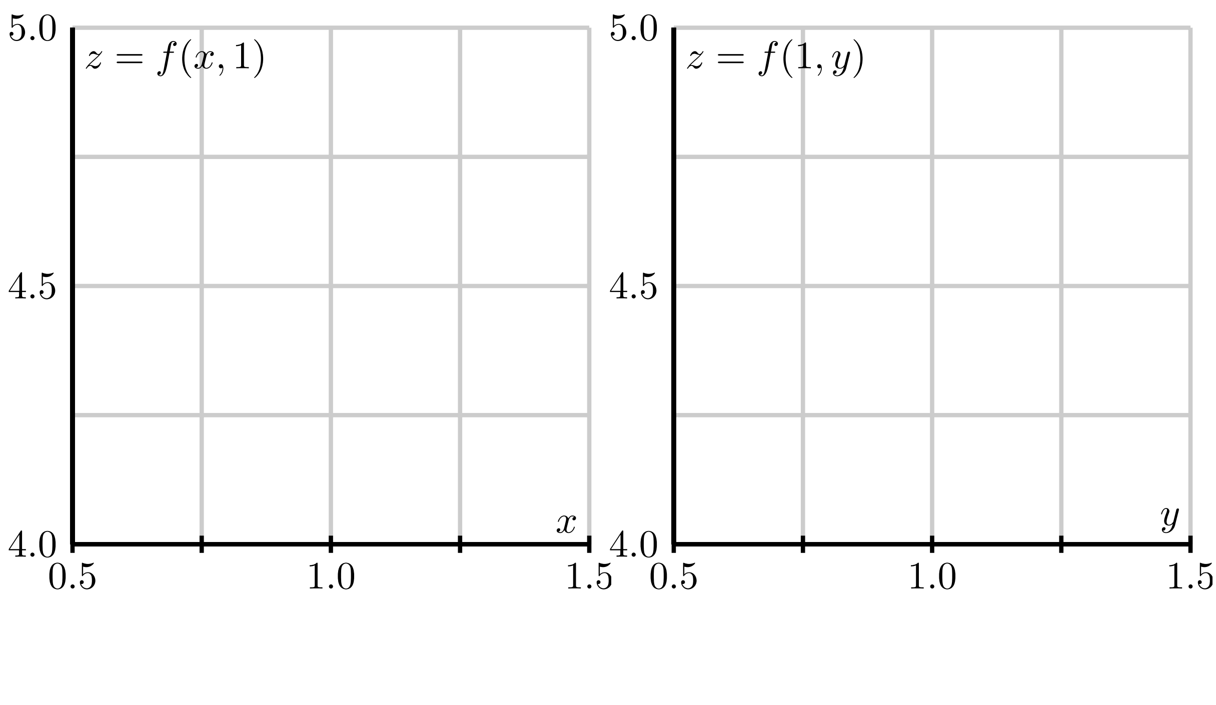
Preview Activity 11.6.1. We want to find the equation of the plane, using the form given in Key Idea 11.6.2, that best describes the surface given by $z = f(x, y) = 6 - \frac{x^2}{2} - y^2$ for input values around $(x_0, y_0) = (1, 1)$. In particular, we will need to find how the values of z_0, a, and b are related to $f(x, y)$.

(a) Find $f(1, 1)$, $f_x(1, 1)$, and $f_y(1, 1)$.

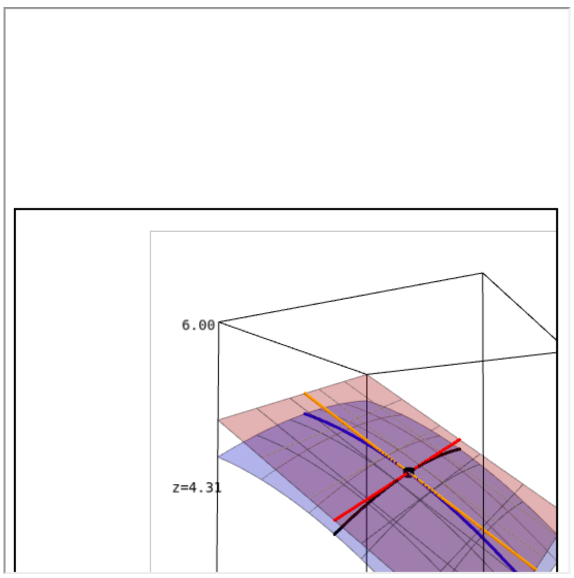
(b) We want our plane to match the height of the surface at $(1, 1)$. For this to happen, the value z_0 from Key Idea 11.6.2 must be the z-coordinate value where the plane intersects the surface. What is the z-coordinate of the point where the tangent plane and the surface intersect?

(c) For the plane to have the same behavior as the surface $z = f(x, y)$ near $(1, 1)$, the plane must match the behavior of the traces given by $x = 1$ and $y = 1$ near this point.

Sketch the traces of $f(x, y) = 6 - \frac{x^2}{2} - y^2$ for $y = y_0 = 1$ and $x = x_0 = 1$ below. Draw the tangent lines to the each of the traces when the appropriate input is 1.



(d) What are the slopes of the tangent lines to the traces that you drew in the previous part? Write a couple of sentences to explain why the tilt of the tangent plane in the x-direction is given by the partial derivative f_x and the tilt of the tangent plane in the y-direction is given by the partial derivative f_y. Your answer should discuss how each of these slopes/partial derivatives relates to the traces of the surface and the plane illustrated below. You should identify whether the red/orange traces along the plane and the black/blue traces along the surface correspond to constant values of x or y.



Standalone

In Context

Embed

(e) Fill in the blanks below with the proper values to give the tangent plane to the graph of $f(x, y) = 6 - x^2/2 - y^2$ at the point $(x_0, y_0) = (1, 1)$.

$$z = z_0 + a(x - x_0) + b(y - y_0) = \_\_\_ + \_\_\_(x - \_\_\_) + \_\_\_(y - \_\_\_).$$

Activity 11.6.2.

(a) Find the equation of the tangent plane to $f(x, y) = x^2y$ at the point $(1, 2)$.

(b) Suppose that the tangent plane to the graph of a continuously differentiable function $z = g(x, y)$ is given in the form

$$z = 5 - 3(x + 2) + (y - 3).$$

Use the equation of the tangent plane to identify a point on the graph as well as a value of g_x and a value of g_y. Be sure to identify at what point(s) you have found the values of the partial derivatives.

Activity 11.6.3. In this activity, we will find the linearization of several different functions that are given in algebraic, tabular, or graphical form.

(a) Find the linearization $L(x,y)$ for the function g defined by

$$g(x,y) = \frac{x}{x^2+y^2}$$

at the point $(1,2)$. Use the linearization to estimate the value of $g(0.8, 2.3)$.

(b) The table below provides a collection of values of the wind chill $w(v,T)$, in degrees Fahrenheit, as a function of wind speed, in miles per hour, and temperature, also in degrees Fahrenheit.

$v\backslash T$	-20	-15	-10	-5	0	5	10
10	-41	-35	-28	-22	-16	-10	-4
15	-45	-39	-32	-26	-19	-13	-7
20	-48	-42	-35	-29	-22	-15	-9
25	-51	-44	-37	-31	-24	-17	-11
30	-53	-46	-39	-33	-26	-19	-12
35	-55	-48	-41	-34	-27	-21	-14

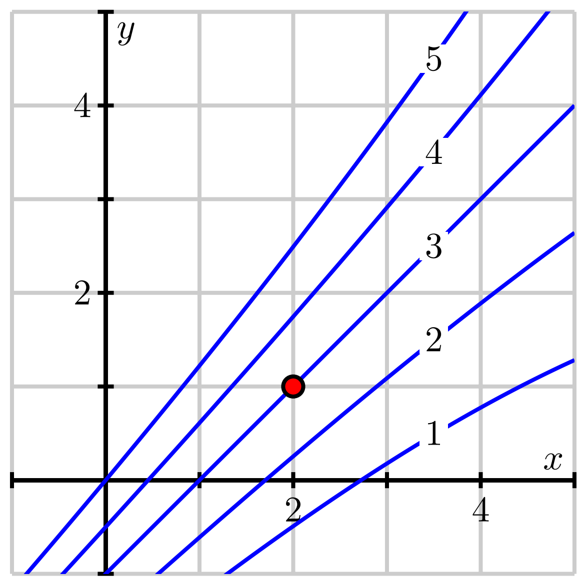
i Use the data to estimate the appropriate partial derivatives of $w(v,T)$ at the point $(20,-10)$.

ii Then find the linearization $L(v,T)$ at the point $(20,-10)$.

iii Use the linearization to estimate $w(10,-10)$, $w(20,-12)$, and $w(18,-12)$.

iv Compare your results to what you obtained in Activity 11.4.5.

(c) The image below gives a contour plot of a continuously differentiable function f. After estimating appropriate partial derivatives, determine the linearization $L(x,y)$ at the point $(2,1)$, and use it to estimate $f(2.2,1)$, $f(2,0.8)$, and $f(2.2,0.8)$.


y
x
4
2
2
4
5
4
3
2
1

Activity 11.6.4.

(a) Suppose that the elevation of a plot of land is given by the function h, where we additionally know that $h(3,1) = 4.35$, $h_x(3,1) = 0.27$, and $h_y(3,1) = -0.19$. Assume that x and y are measured in miles in the east and north directions, respectively, from $(0,0)$.

Your GPS device says that you are currently at the point $(3,1)$. However, you know that the coordinates are only accurate to within 0.2 miles; that is, $dx = \Delta x = 0.2$ and $dy = \Delta y = 0.2$. Estimate the uncertainty in your elevation using the linearization of h for the input (3,1).

(b) The pressure, volume, and temperature of an ideal gas are related by the equation

$$P = P(T,V) = 8.31\frac{T}{V},$$

where P is measured in kilopascals, V in liters, and T in kelvin. Find the pressure when the volume is 12 liters and the temperature is 310 K. Use the linearization of P at the point $(310,12)$ to estimate the change in the pressure when the volume increases to 12.3 liters and the temperature decreases to 305 K.

(c) Use the table of values for the wind chill $w(v,T)$, in degrees Fahrenheit, as a function of temperature, also in degrees Fahrenheit, and wind speed, in miles per hour provided in part 11.6.3.b for this part. Suppose your anemometer[a] says the wind is blowing at 25 miles per hour and your thermometer[b] shows a reading of $-15°$ degrees. However, you know your thermometer is only accurate to within $2°$ degrees and your anemometer is only accurate to within 3 miles per hour. What is the wind chill based on your measurements? Estimate the uncertainty in your measurement of the wind chill.

[a]An instrument for measuring wind speed.
[b]An instrument for measuring the temperature.

11.7 The Multivariable Chain Rule

Preview Activity 11.7.1. Your self-driving car company, *Steer Clear*, is doing well and almost ready to launch its first car. Some of your engineers have reported that the car has problems when the air intake encounters a large quantity of large particulate matter (sand, dust, large pollen, smog, etc.) in the air. To fix this issue, you have created a new type of filter that uses a sophisticated mesh and gravity to filter out large particulate matter. However, this new type of filter gets clogged if exposed to too much large particulate matter too quickly. To determine the viability of this filter, you must measure the rate at which your car's intake will be exposed to large particulate matter per unit time.

You consult a friend Alex who does atmospheric modeling of large particulate matter in your area. Alex created a function that describes the amount of large particulate matter in the air in terms of location relative to her lab. This function is expressed algebraically as $P(x, y) = 10 - \frac{1}{2}x^2 - \frac{1}{5}y^2$. Here x is the distance east/west from Alex's lab and y is the distance north/south from her lab. Both distances are measured in kilometers and P is measured in parts per million.

Using the self-driving feature of your car, you specify that your car will move along a test course that can be described as

$$\vec{r}(t) = \langle x(t), y(t)\rangle = \langle 2 - t^2, t^3 + 1\rangle,$$

where the x and y coordinates are the same as measured by Alex's function and t is measured in minutes.

(a) Substitute the path component functions $x(t)$ and $y(t)$ into the expression for $P(x, y)$ but ***do not simplify*** the resulting expression for $P(t)$. This gives a one-variable function that describes the amount of large particulate matter encountered at each location on your test course as a function of *time*.

(b) Use derivative rules from single-variable calculus to find $P'(t) = \frac{dP}{dt}$. ***Do not simplify this expression.*** Note that this is *not* a partial derivative because $P(t)$ only has *one* dependent variable.

(c) Alex has told you that the amount of large particulate matter in the air changes in the y-direction (north/south) because of the proximity to industrial pollution from a factory complex. Also, the amount of large particulate matter in the air changes in the x-direction (east/west) because of tree pollen from a species of tree in a nearby forest. You would like to understand how the rate of large particulate matter encountered on the car's drive is split into these two factors.

Alex suggests that you measure $\frac{dP}{dt}$ at a location $(x, y) = (a, b)$ by taking the derivative of the linearization for P at (a, b). The linearization of P at (a, b) is given by

$$L(x, y) = P(a, b) + P_x(a, b)(x - a) + P_y(a, b)(y - b)$$

Remember that $x = x(t)$ and $y = y(t)$ are being considered as functions of time in this scenario.

Write a couple of sentences to explain why

$$\frac{dP}{dt} \approx \frac{dL}{dt} = 0 + P_x(a,b)\frac{dx}{dt} + P_y(a,b)\frac{dy}{dt} \tag{11.7.1}$$

and how $P_x(a,b)\frac{dx}{dt}$ and $P_y(a,b)\frac{dy}{dt}$ measure the rates of change in pollution coming from tree pollen and industrial pollution, respectively.

(d) Use the formulas provided at the beginning of the Preview Activity for $P(x,y)$, $x(t)$, and $y(t)$ to compute each of the following. Then substitute your expressions into Alex's approximation given in equation (11.7.1). ***Do not simplify this expression.***

$$P_x(a,b) \qquad P_y(a,b) \qquad \frac{dx}{dt} \qquad \frac{dy}{dt}$$

(e) Compare your results for part (b) and part (d) and write a couple of sentences that describe how each part of your answer for part (d) corresponds to different parts of part (b). Remember your work for part (b) is completely in terms of t so you will need to translate some terms from t into its meaning in other quantities.

Activity 11.7.2. In the following questions, we apply the chain rule in several different contexts.

(a) Suppose $z(x, y) = x^2 + xy^3$. In addition, suppose that x and y are restricted to points that move around the plane by following a circle of radius 2 centered at the origin that is parameterized by

$$x(t) = 2\cos(t), \quad \text{and} \quad y(t) = 2\sin(t).$$

i. Use the chain rule to find the instantaneous rate of change $\frac{dz}{dt}$.

ii. Substitute $x(t)$ for x and $y(t)$ for y in the rule for z to write z in terms of t and calculate $\frac{dz}{dt}$ using only techniques from single-variable calculus.

iii. Write a couple of sentences comparing your answers above.

(b) Suppose that the temperature on a metal plate is given by the function T with

$$T(x, y) = 100 - (x^2 + 4y^2)$$

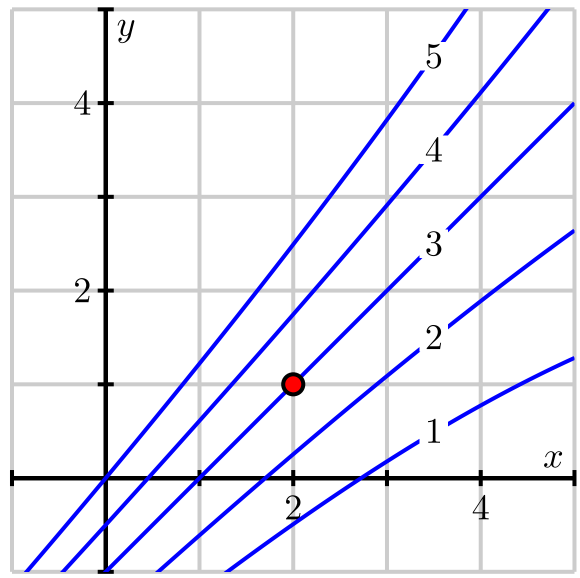
where the temperature is measured in degrees Fahrenheit and x and y are each measured in feet.

i. Find T_x and T_y. What are the units on these partial derivatives?

ii. Suppose an ant is walking along the x-axis at the rate of 2 feet per minute toward the origin. When the ant is at the point $(2, 0)$, what is the instantaneous rate of change in the temperature dT/dt that the ant experiences. Include units on your response.

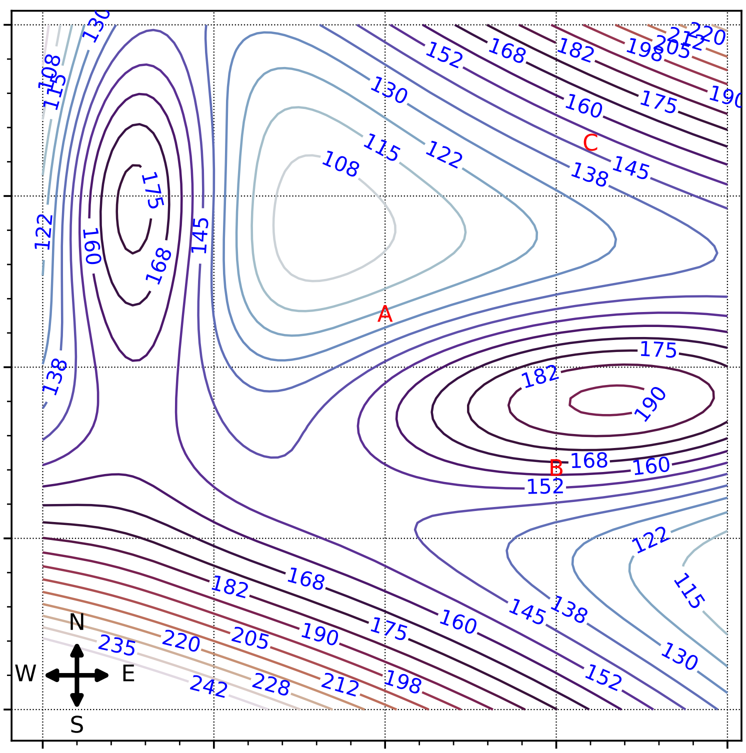
iii. Suppose instead that the ant walks along the ellipse $\vec{r}(t) = \langle 6\cos(t), 3\sin(t)\rangle$, where t is measured in minutes. Use the multivariable chain rule to find $\frac{dT}{dt}$. What does this tell you about the path along which the ant is walking?

(c) Suppose that you are walking along a surface whose elevation is given by a function f. Some contours of f are as shown in the image below. Furthermore, suppose that if you consider how your location corresponds to points in the xy-plane, you know that when you pass the point $(2, 1)$, your velocity vector is $\vec{v} = \langle -1, 2\rangle$. Estimate the rate of change df/dt when you pass through $(2, 1)$.


y
x
4
2
2
4
5
4
3
2
1

11.8 Directional Derivatives and the Gradient

Preview Activity 11.8.1. Below is a contour plot showing the elevations for a region of a nearby park. We will be referring to $h(x, y)$ as the function of two variables that gives the elevation as a function of x-coordinate (location in the east-west or horizontal direction) and y-coordinates (location in the north-south or vertical direction).



(a) Using the contour plot and treating the elevation as a multivariable function $h(x, y)$, state whether each of the following is positive, negative, or zero. Write a sentence to justify your reasoning.

i. $h_x(A)$

ii. $h_y(A)$

iii. $h_x(B)$

iv. $h_y(B)$

v. $h_x(C)$

vi. $h_y(C)$

(b) Suppose you are standing at point B. Do you expect the elevation to increase, decrease, or remain constant if you take a step in the northeast direction? Write a sentence to explain your reasoning.

(c) Suppose you are standing at point B. Do you expect the elevation to increase, decrease, or remain constant if you take a step in the southeast direction? Write a sentence to explain your reasoning.

(d) Suppose you are standing at point A. Do you expect the elevation to increase, decrease, or remain constant if you take a step in the southwest direction? Write a sentence to explain your reasoning.

(e) Suppose you are standing at point C. In what direction would you take a step to move in the steepest downhill direction? Write a sentence to explain your reasoning.

(f) Suppose you are standing at point A. In what direction would you take a step to move in the steepest uphill direction? Write a sentence to explain your reasoning.

(g) Suppose you are standing at point B. Rank the following directions in order of steepness (from most steep and uphill to level to most steep downhill): northeast, north, east, west, south

Activity 11.8.2. In this activity, we will use equation (11.8.2) to calculate and interpret directional derivatives for the function $f(x,y) = 3xy - x^2y^3$.

(a) Calculate $f_x(x,y)$ and $f_y(x,y)$.

(b) Use equation (11.8.2) to determine $D_{\hat{\imath}}f(x,y)$ and $D_{\hat{\jmath}}f(x,y)$. Write a couple of sentences to describe what familiar functions $D_{\hat{\imath}}f$ and $D_{\hat{\jmath}}f$ are.

(c) Use equation (11.8.2) to find the derivative of f in the direction of the vector $\vec{v} = \langle 2, 3\rangle$ at the point $(1,-1)$.

(d) Find the derivative of f in the direction of the vector $\vec{v} = \langle 4, 6\rangle$ at the point $(1,-1)$.

(e) Use equation (11.8.2) to find the derivative of f in the direction of the vector $\vec{v} = \langle -2, -3\rangle$ at the point $(1,-1)$. Write a couple of sentences to explain why this result is different from your answer to the previous two tasks, even though the direction vectors are parallel.

Activity 11.8.3. Let $f(x,y) = \frac{1}{3}(x^2 - y^2)$. Some contours for this function are shown in Figure 11.8.13.

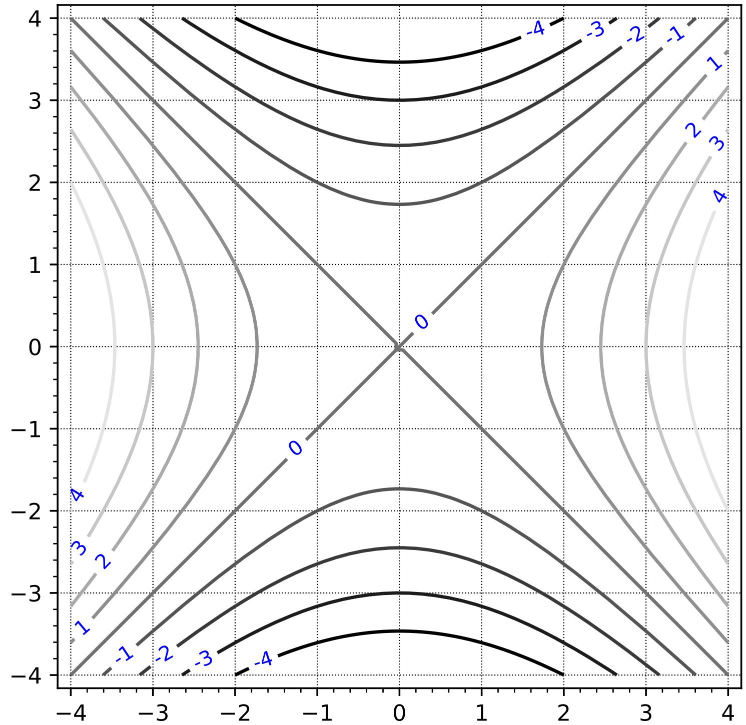


Figure 11.8.13 A contour plot of $f(x,y) = \frac{1}{3}(x^2 - y^2)$

(a) Find the gradient $\nabla f(x,y)$.

(b) For each of the following points (x_0, y_0), evaluate the gradient $\nabla f(x_0, y_0)$ and sketch the gradient vector with its tail at (x_0, y_0). Some of the vectors are too long to fit onto the plot. To draw them to scale, you should scale each vector by a factor of $1/2$.

- $(x_0, y_0) = (2, 0)$
- $(x_0, y_0) = (0, 2)$
- $(x_0, y_0) = (2, 2)$
- $(x_0, y_0) = (2, 1)$
- $(x_0, y_0) = (-3, 2)$
- $(x_0, y_0) = (-2, -4)$
- $(x_0, y_0) = (0, 0)$

(c) Write a few sentences about how the direction of the gradient at each of these points is related to the the contour passing through that point.

(d) Does the output of f increase or decrease in the direction of $\nabla f(x_0, y_0)$? Use examples from the points above to write a couple of sentences that justify your answer.

Activity 11.8.4. For this activity, we will look at the value of the directional derivative for temperature as a function of location. When looking at a plot of thermoclines (locations with the same temperature), you notice that your town is exactly on the thermocline labeled as 70 degrees, shown as the point P on Figure 11.8.15

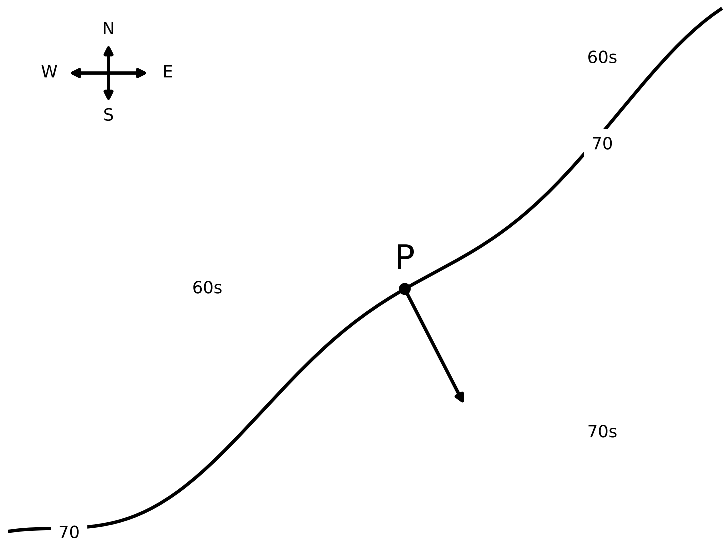


Figure 11.8.15 A plot of the thermocline corresponding to 70 degrees going through the point P

(a) Write a few sentences about why the gradient at P goes in the southeast direction and not the northwest (which is also perpendicular to the thermocline/ level curve at P).

(b) Draw a vector in each of the following directions starting at P and state whether the directional derivative will be positive/negative/0 in each of these directions. You should write a couple of sentences for each direction vector about how the sign of the directional derivative is related to the angle between the direction and the gradient vectors.

i. West

ii. South

iii. Northeast

(c) Write a few sentences about why any direction moving up and left will have negative rate of change for the temperature in the direction given.

(d) Explain what direction you would need to move in to have the greatest positive rate of change for the temperature. Write a couple of sentences to relate your answer to the gradient using Equation (11.8.5).

Activity 11.8.5. We will look algebraically calculating and explaining the vector properties of the gradient using the function $f(x, y) = xy - 2y^2 + \frac{x^2}{2} + 2y - x$ at the input $(1, 1)$.

(a) Calculate the gradient of f

(b) Calculate $\nabla f(1, 1)$.

(c) Calculate $D_{\vec{u}} f(1, 1)$ for each of the following vectors:

i. $\vec{u} = \langle -1, 0 \rangle$

ii. $\vec{u} = \langle \dfrac{3}{5}, -\dfrac{4}{5} \rangle$

iii. $\vec{u} = \langle -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \rangle$

(d) What direction will correspond to the greatest rate of increase for f at the input $(1, 1)$? Find the value of the directional derivative in this direction.

Activity 11.8.6. This activity considers directional derivatives and gradients for a function that measures elevation. Suppose you are hiking in a foggy park and can only see a few feet in front of you. There is nothing blocking you from walking in any particular direction, but because of the fog you cannot see where the highest point on the mountain is. You want to try to find the top of the mountain, but you don't have a map, trail, or line of sight to other landmarks. You do have a compass, which works in the fog. This allows you to identify the cardinal directions.

In order to use calculational tools from multivariable calculus, you overlay an xy-coordinate system on the park. The x-coordinate measures east/west location, with eastward movement associated with increasing x-values. The y-coordinate measures north/south location, with northward movement associated with increasing y-values. The elevation in meters above sea level of the park's terrain at the location (x, y) is given by a function $h(x, y)$.

(a) Let P_1 be your current location in the foggy park. You use your compass to find the east and north directions. At P_1, you find that the ground rises 1 meter per 50 meters traveled to the east and the ground rises 2.5 meters per 50 meters traveled to the north.

Use this information to find $\nabla h(P_1)$.

(b) Use your answer to the previous part to say what direction is "uphill" at P_1 and state the rate of elevation increase in this direction.

(c) You decide to walk uphill from your location P_1 in order to try to find the top of the mountain. After walking in the same direction for a while, you are at a point P_2 and notice that you are no longer walking in the steepest direction. You again locate east and north and measure the steepness of the mountain in these directions. You find that the ground rises 1.5 meters per 75 meters traveled to the east and the ground goes down 0.5 meters per 100 meters traveled to the north.

Use this new information to calculate $\nabla h(P_2)$, find the uphill direction, and find how steep the mountain is in the uphill direction at P_2.

(d) Suppose you continue this method of walking in the uphill direction for a while, finding the new uphill direction, and walking in the new uphill direction. Do you think you must eventually find the top of the mountain? How you will know that you have reached the top of the mountain? Remember that you can't see very far in front of you. Write a few sentences to explain your reasoning.

11.9 Higher Dimensions

Preview Activity 11.9.1.

(a) State the equation and shape of the level curves for the function $f(x, y) = x^2 + y^2$ for the values $-2, -1, 0, 1, 2$.

(b) State the equation and shape of the level curves for the function $g(x, y) = x^2 - y^2$ for the values $-2, -1, 0, 1, 2$.

(c) For a constant k and a function f, the **level set** is the set of points P for which $f(P) = k$. The remainder of this Preview Activity asks you to consider level sets of some functions of three variables. We use the term "level set" here because the set of inputs for a function of three or more variables that gives a particular output value will likely not be a curve.

State the equation and shape of the level sets for the function $f(x, y, z) = x^2 + y^2 + z^2$ for $-2, -1, 0, 1, 2$.

(d) State the equation and shape of the level sets for the function $g(x, y, z) = x^2 + y^2 - z^2$ for $-2, -1, 0, 1, 2$. These level set values should give three different types of surfaces.

(e) State the equation and shape of the level sets for the function $h(x, y, z) = x + y - z$ for $-2, -1, 0, 1, 2$.

Activity 11.9.2. In this activity, we will look at level surfaces created by two three-variable functions and consider the direction in which the output is increasing or decreasing as fast as possible.

(a) Let $f(x, y, z) = 2x - y + z$.

(i) Write a couple of sentences to describe shape and characteristics will the level surfaces of f. Be sure to include all possibilities for different types of level surfaces. Compare what is the same or different about these level surfaces and relate the differences to the level value, k.

(ii) If you are at the point $(1, 1, -1)$, in what direction should you change the input of f to see the greatest increase in the output of f? You should think about how a change in each variable will change the output of f and talk about this in your reasoning.

(b) Let $g(x, y, z) = x^2 + y^2 + z^2$.

(i) Write a couple of sentences to describe shape and characteristics will the level surfaces of g. Be sure to include all possibilities for different types of level surfaces. Compare what is the same or different about these level surfaces and relate the differences to the level value, k.

(ii) If you at the point $(2, 0, -2)$, in what direction should you change the input of g to see the greatest increase in the output of g? You should think about how a change in each variable will change the output of g and talk about this in your reasoning.

Activity 11.9.3. In this activity, we will look at the calculation and interpretation of the gradient for $f(x, y, z) = x + y - z$ and $g(x, y, z) = z - x^2 - y^2$.

(a) Calculate ∇f and ∇g.

(b) Sketch the level surfaces of f for the values $k = \{-2, -1, 0, 1, 2\}$. Write a few sentences about the shape of each of these level surfaces and describe how the level surfaces change in terms of the value of k.

(c) Write a few sentences about how the direction and magnitude of ∇f is related to the level surfaces from the previous part.

(d) Sketch the level surfaces of g for the values $k = \{-2, -1, 0, 1, 2\}$. Write a few sentences about the shape of each of these level surfaces and describe how the level surfaces change in terms of the value of k. You may find it helpful to notice that each of the level surfaces can be expressed with z as a function of x and y.

(e) Write a few sentences about how the direction and magnitude of ∇g is related to the level surfaces from the previous part.

11.10 Optimization

Preview Activity 11.10.1. In this activity, we will use a function that takes a two-dimensional location as its input and outputs the elevation of a surface at that point to explore what properties a point corresponding to a local maximum or local minimum of a function of two variables must have. Similar to Activity 11.8.6, imagine that you are hiking in a foggy park and cannot see anything more than a few feet in front of you. There is nothing blocking you from walking in any particular direction, but because of the fog, you cannot see where the highest point on the mountain is. You want to try to find the top of the mountain, but you don't have a map to consult or trail to follow or even a line of sight to other landmarks. You do have a compass that works in the fog, which you can use to determine directions. You also have a level that can be placed on the ground at your feet to measure how steep the change in elevation is in a particular direction. In other words, the level can measure the directional derivative at a location. Conceptually, our goal in this preview activity is to understand how the measurements you can take with your level allows you to identify when you have reached the top of the mountain.

(a) In Activity 11.8.6, you saw how you can move toward the top of the mountain by taking steps in the "uphill" direction, which is the direction of greatest rate of increase for elevation. Suppose you are standing at the top of the mountain, which corresponds to a local maximum for elevation, and you set the level at your feet while facing east. Will the level measure that the elevation is increasing, decreasing, or constant? Write a couple of sentences to justify your answer.

(b) If you are at the top of the mountain and set the level at your feed while facing south, will the level measure that the elevation is increasing, decreasing, or constant? Write a couple of sentences to justify your answer.

(c) Write a few sentences to explain why your level *must* show a zero rate of change in *every* direction at the top of the mountain.

(d) If you took a small step to the west from the top of the mountain, should your level measurement to the west at your new location indicate that the elevation is increasing, decreasing, or constant? Explain your reasoning.

(e) If you took a small step in *any direction* from the top of the mountain, should your level measurement in the direction you took the step at your new location indicate that the elevation is increasing, decreasing, or constant? Explain your reasoning.

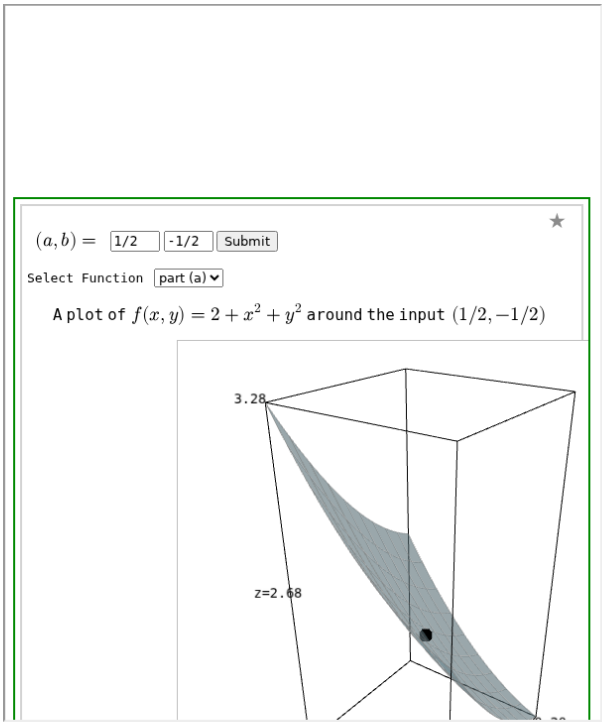
(f) Now imagine that you are at the *lowest* point in the foggy park. Write a couple of sentences to explain why your level *must* show a zero rate of change in *every* direction.

(g) If you took at step in *any direction* from the lowest point in the park, should your level measurement at your new location in the the direction in which you took the step indicate that the elevation is increasing, decreasing, or constant? Explain your reasoning.

(h) Is it possible on your hike in the fog that you find a point where your level shows constant elevation in every direction but you are *not* at the top of the mountain or lowest point on the mountain? Explain what the the terrain would look like at this type of location.

Activity 11.10.2. For each of the functions below, do the following:

1. Find all critical points for the function.
2. Use Figure 11.10.7 to plot the surface near each critical point. Enter the coordinates of your critical point as (a, b) and the function you are using for $f(x, y)$.
3. Use your plots of the surfaces $z = f(x, y)$ near your critical point to decide whether each critical point is a local maximum, a local minimum, or not an extreme point.



Standalone

In Context

Embed

Figure 11.10.7 Plots of $z = f(x, y)$ around each of the critical points

(a) $f(x, y) = 2 + x^2 + y^2$

(b) $f(x, y) = 2 + x^2 - y^2$

(c) $f(x, y) = 2x - x^2 - \frac{1}{4}y^2$

(d) $f(x, y) = |x| + |y|$

(e) $f(x, y) = 2xy - 4x + 2y - 3$

Activity 11.10.3. For each of the functions given below:

1. Find the critical points of the function
2. Compute the discriminant D and evaluate D for each critical point.
3. Use the Second Partial Derivatives Test to classify the critical points

(a) $f(x,y) = 3x^3 + y^2 - 9x + 4y$

(b) $f(x,y) = xy + \frac{2}{x} + \frac{4}{y}$

(c) $f(x,y) = x^3 + y^3 - 3xy$

Activity 11.10.4. While the quantity of a product demanded by consumers is often a function of the price of the product, the demand for a product may also depend on the price of other products. For instance, the demand for blue jeans at a clothing store may be affected not only by the price of the jeans themselves, but also by the price of khakis.

Suppose we have two goods whose respective prices are p_1 and p_2. The demand for these goods, q_1 and q_2, depend on the prices by the following relations:

$$q_1 = 150 - 2p_1 - p_2 \tag{11.10.1}$$

$$q_2 = 200 - p_1 - 3p_2 \tag{11.10.2}$$

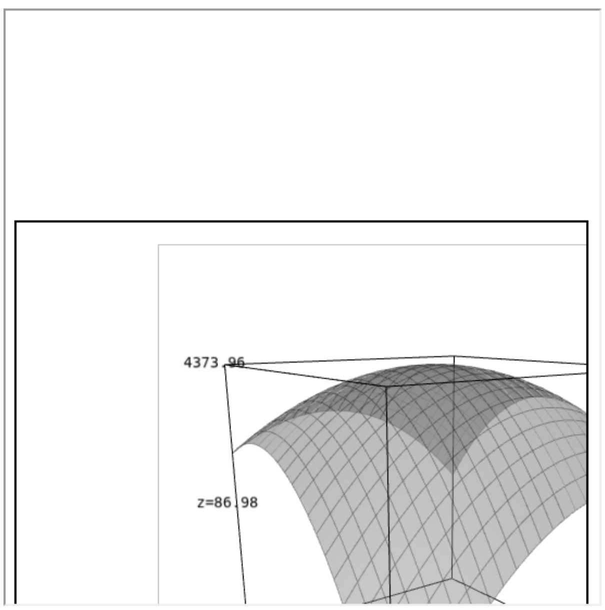
The seller wants to set the prices p_1 and p_2 in order to maximize revenue. We will assume that the seller meets the full demand for each product. Thus, if we let R be the revenue obtained by selling q_1 items of the first good at price p_1 per item and q_2 items of the second good at price p_2 per item, we have

$$R = p_1q_1 + p_2q_2.$$

We can then write the revenue as a function of just the two variables p_1 and p_2 by using Equations (11.10.1) and (11.10.2), giving us

$$\begin{aligned} R(p_1, p_2) &= p_1(150 - 2p_1 - p_2) + p_2(200 - p_1 - 3p_2) \\ &= 150p_1 + 200p_2 - 2p_1p_2 - 2p_1^2 - 3p_2^2. \end{aligned}$$

A graph of R as a function of p_1 and p_2 is shown in below.



Standalone

In Context

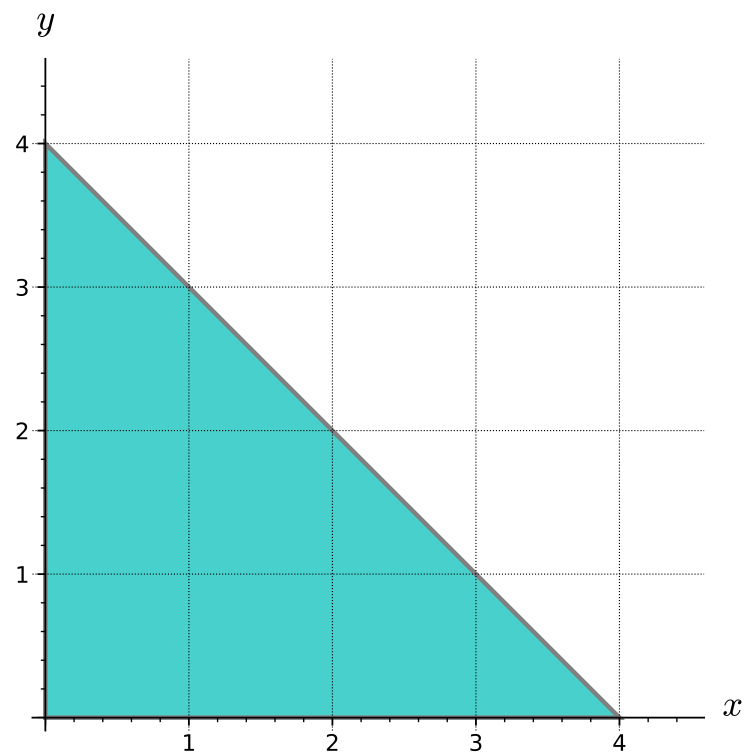
Embed

(a) Find all critical points of the revenue function R.

(b) Apply the Second Partial Derivatives Test to determine the type of any critical point(s).

(c) Where should the seller set the prices p_1 and p_2 to maximize the revenue?

Activity 11.10.5. Let $f(x, y) = x^2 - 3y^2 - 4x + 6y$ with triangular domain R whose vertices are at $(0, 0)$, $(4, 0)$, and $(0, 4)$. The domain R is illustrated below. In this activity, you will carry out the steps to find the absolute maximum and absolute minimum of f on R.



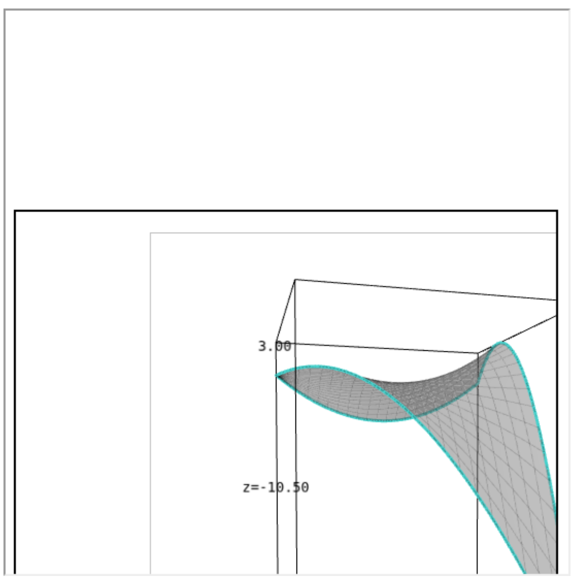
(a) Find all of the critical points of f in R.

(b) Parameterize the edge of R that is on the x-axis and find the critical points of f on that edge.

(c) Parameterize the edge of R that is on the y-axis and find the critical points of f on that edge.

(d) Parameterize the diagonal edge of R and find the critical points of f on that edge.

(e) Find the absolute maximum and absolute minimum values of f on R. Write a couple of sentences to describe how the surface plot of f shown below illustrates your results.



Standalone

In Context

Embed

11.11 Constrained Optimization: Lagrange Multipliers

Preview Activity 11.11.1. A shipping company places a restriction on packages so that the sum of the package's length and girth must be at most 12 feet. (The girth is the perimeter of the smallest end of the package.) Our goal is to find the largest possible volume of a rectangular package with a *square* end that can be sent by this shipping company.

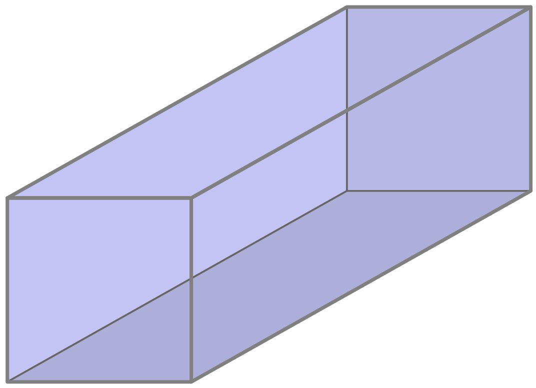


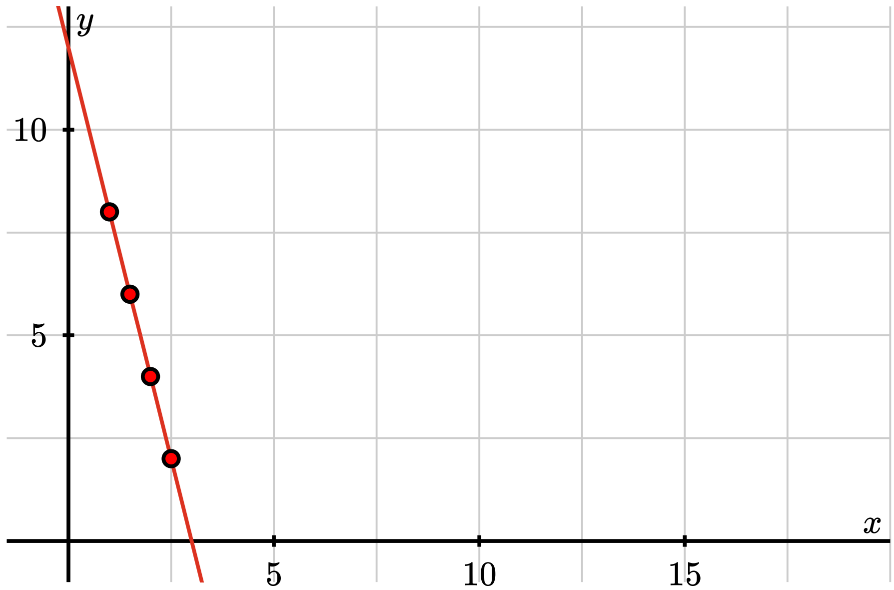
Figure 11.11.1 A shipping box with square end

Let x be the length in feet of the side of one square end of the package and y be the length in feet of the package. This allows us to define a function that gives the volume of the package as $f(x, y) = x^2y$ cubic feet. The girth of the package is $4x$, so we can define $g(x, y) = 4x + y$ to be the sum of the girth and length. The domain on which we wish to maximize f is all points for which $x \geq 0$, $y \geq 0$, and $g(x, y) \leq 12$. A moment of reflection should confirm that any package with $g(x, y) < 12$ cannot maximize the volume, as increasing either x or y so that $g(x, y)$ remains at most 12 will only lead to a larger volume.

Thus, we say that we wish to maximize $f(x, y) = x^2y$ subject to the constraint $g(x, y) = 4x + y = 12$. We call f the objective function since it is our goal to find extreme values for this function. We call g our contraint function because we are constrainted (limited) to a specific value of this function.

(a) Compute ∇f and evaluate the gradient at the points $P_1 : (1, 8)$, $P_2 : (1.5, 6)$, $P_3 : (2, 4)$, and $P_4 : (2.5, 2)$.

(b) The plot below shows the graph of $g(x) = 12$ as well as the four points for which you evaluated ∇f in the previous part. Sketch ∇f at each of the four points. You may not have enough space to sketch get the magnitude of each gradient vector correct, but you should accurately depict the direction of each of the four vectors.

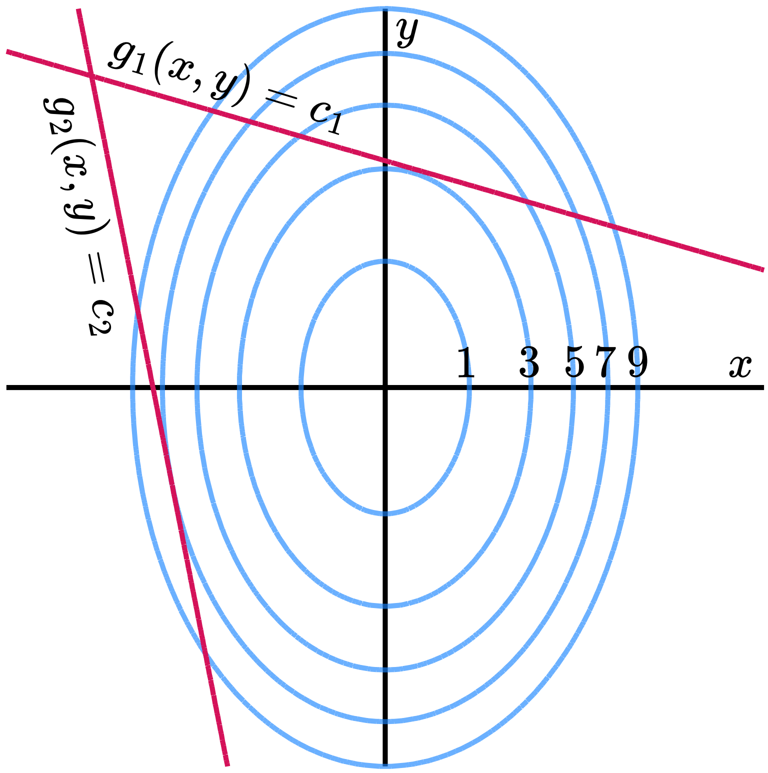


(c) Compute ∇g. Describe the geometric relationship between ∇g and the line $g(x, y) = 12$ plotted in the previous part and why they have this relationship.

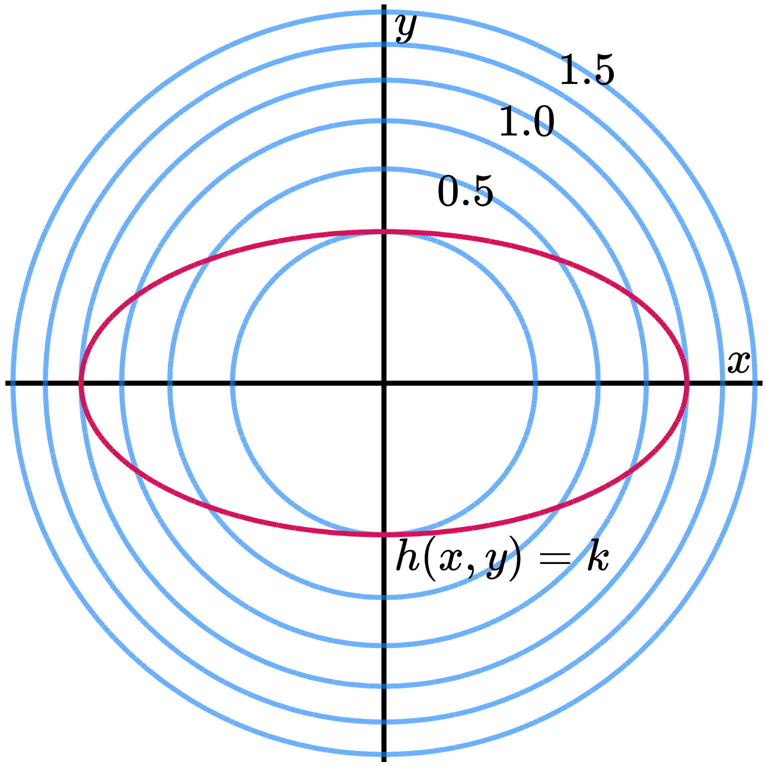
(d) Recall that the gradient always points in the direction of steepest ascent. However, in this *constrained* optimization setting, the domain that we allowed to consider for f is only those points for which $g(x, y) = 12$. Thus, we may not be able to move in the direction of the gradient while staying within the domain. Look back at the four points at which you sketched ∇f in part (b). For each point, identify which direction you could move *along the line* to increase the value of f. What is the relationship between ∇f and ∇g at a point where you cannot move along the line while increasing the value of f?

Activity 11.11.2.

(a) Below is the contour plot of a function $f(x, y)$. Also shown are two lines, which are constraint curves. Use the plot to identify the extreme value(s) of $f(x, y)$ subject to the constraint $g_1(x, y) = c_1$ and the point(s) at which these extrema occur. Repeat this for the constraint $g_2(x, y) = c_2$. Write a sentence to explain your reasoning.



(b) The circular curves in the graph below are contours of a function $T(x, y)$. Use the plot to identify the extreme value(s) of $T(x, y)$ subject to the constraint $h(x, y) = k$, which is also shown in the plot. Also identify the points at which the extreme value(s) occur and write a sentence to explain your reasoning.



y
1.5
1.0
0.5
x
h(x,y) = k

Activity 11.11.3. A cylindrical soda can holds about 355 cc of liquid. In this activity, we want to find the dimensions of such a can that will minimize the surface area, since this will minimize the material used to construct this can. For the sake of simplicity, assume the can is a perfect cylinder.

(a) What are the variables in this problem? Based on the context, what restriction(s), if any, are there on these variables? You will likely find it useful to draw a picture of your cylindrical can and label your variables.

(b) What quantity do we want to optimize in this problem? What equation describes the constraint? You need to decide which of these functions plays the role of the objective function, f, and which plays the role of the constraint function, g, in our discussion of Lagrange multipliers.

(c) Use your choices for the objective and constraint equations to set up a system of equations in terms of your variables and λ corresponding to the Lagrange Multipliers. Find λ and the values of your variables that satisfy Equation (11.11.1) in the context of this problem.

(d) Determine the dimensions of the soda can that give the desired solution to this constrained optimization problem.

Activity 11.11.4. Use the method of Lagrange multipliers to find the dimensions of the least expensive rectangular packing crate with a volume of 240 cubic feet when the material for the top costs \$2 per square foot, the bottom costs \$3 per square foot, and the sides cost \$1.50 per square foot.

12 Multiple Integrals

12.2 Double Riemann Sums and Double Integrals over Rectangles

Preview Activity 12.2.1.

(a) A plot of f for inputs in the interval $[0,2]$ is shown in Figure 12.2.1. Break the interval $[0,2]$ into four equally sized subintervals and draw the rectangles that would be used to construct a Riemann sum to approximate the area under f on the interval $[0,2]$. You can use whichever point you want on each subinterval to evaluate the height the of the rectangles.

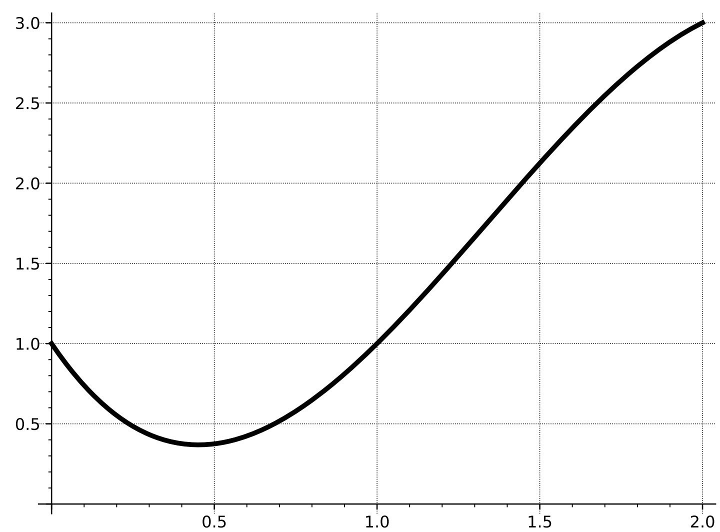


Figure 12.2.1 A continuous function on $[0,2]$

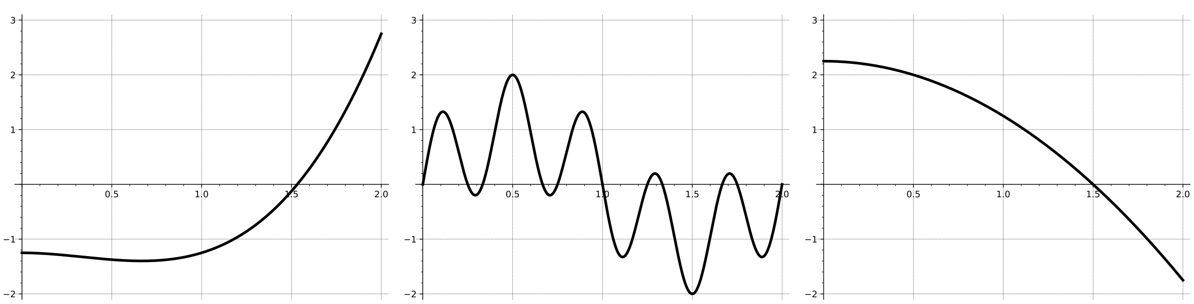
(b) Estimate the heights of the rectangles used in your Riemann sum above to estimate $\int_0^2 f(x)\ dx$. Write a couple of sentences about why you think your estimate for the definite integral of f is either an overestimate, an underestimate, or close to the true value.

(c) Recall that the definite integral is defined as

$$\int_a^b f(x)\ dx = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

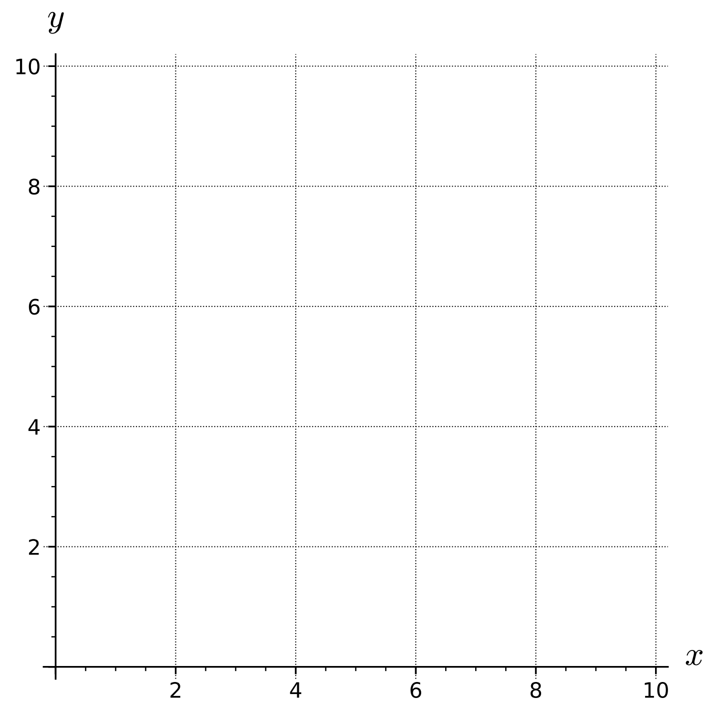
Explain why it doesn't matter what method (left endpoint, right endpoint, midpoint, etc.) you use for selecting which point is evaluated on each of the subintervals in the definition of the definite integral.

(d) For each of the functions plotted below, determine if the definite integral over the region shown will be positive, negative, or zero and write a sentence to justify your answer.



Activity 12.2.2. In this activity, we will go through the first two steps of the classic calculus approach to compute the definite integral of $f(x, y) = 100 - x^2 - y^2$ on the rectangular domain $R = [0, 8] \times [2, 6]$. Remember that we are trying to measure the volume below the graph of f over the region R, so we will start with estimating this volume.

(a) To understand the numerical calculations involved in the classic calculus approach for a double integral, it is most important to understand the region of integration. Thus, we will not look at a graph of $z = f(x, y)$. Instead, we will stay focused in the xy-plane. On the axes below, outline the rectangular region R that corresponds to the region of integration.



(b) Because all of the regions and subregions we are considering in this section are rectangles, we can break up the x and y coordinates into pieces separately. For this activity, break the interval of x-coordinates into *four* equally-sized subintervals and break the interval of y-coordinates into *three* equally-sized subintervals.

What is the length of each subinterval in the x-direction? How about in the y-direction?

(c) Let S_i be the i-th subinterval for x. We want to state the endpoints of each of the S_i. The first subinterval, S_1, will go from x_0 to x_1, the second subinterval, S_2, will go from x_1 to x_2, the third subinterval, S_3, will go from x_2 to x_3, and the fourth subinterval, S_4, will go from x_3 to x_4.

Give the values for x_0, x_1, x_2, x_3, and x_4 and add these as tick marks on the x-axis of the graph in part (a) to make sure your subintervals are equally sized.

(d) Let T_i be the i-th subinterval for y. Give the values for y_0, y_1, y_2, and y_3 and add these as tick marks on the y-axis of the graph in part (a) to make sure your subintervals are equally sized.

(e) We will use the subintervals in the x- and y-coordinates to specify smaller rectangles into which R is divided to compute an approximation. Let R_{ij} be the rectangle corresponding to $S_i \times T_j$.

i. How many smaller rectangles are there in this partition?

ii. Outline each of the smaller rectangles on the graph in part (a) and label each rectangle as either $R_{11}, R_{12}, \ldots$.

iii. Since each smaller rectangle R_{ij} has the same area, let ΔA denote the area of each of these smaller rectangles. What is ΔA?

(f) To find the volume of the rectangular prisms over each R_{ij}, we must pick a point in each subrectangle at which to evaluate f, which will be the height of the rectangular prism. For this activity, we will use the upper-right corner of each subrectangle as the designated point.

State the point at the upper right of each smaller rectangle and evaluate f at each of these points.

(g) Write a sentence about why the volume of each rectangular prism used for this approximation is

$$f(x_i, y_j)\Delta A.$$

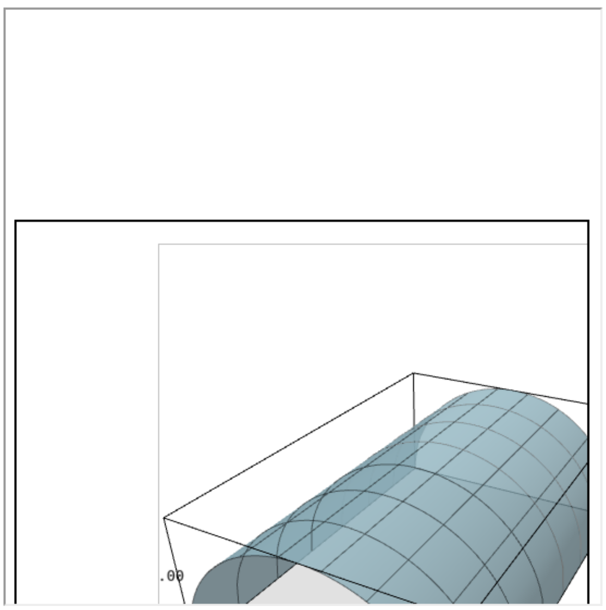
Remember that x_i and y_j are the points from parts c and d of this activity. Write a couple sentences about how you would find an approximation of the volume under the surface $z = f(x, y)$ over the region R. (Do not do calculation, but rather explain what calculation is being done.)

(h) We chose the upper-right point of each subrectangle R_{ij} to find the height of each rectangular prism used in the approximation. Write a sentence or two about whether you think the upper right point provides an overestimate, an underestimate, or approximately the average value for f on each R_{ij}. Explain how this suggests that your estimate for the volume under the surface $z = f(x, y)$ over the region R is either an overestimate, an underestimate, or approximately the correct value.

Activity 12.2.3. Let $f(x,y) = \sqrt{4-y^2}$ on the rectangular region $R = [1,7] \times [-2,2]$. Suppose that we partition $[1,7]$ into 3 subintervals of equal length and $[-2,2]$ into 2 subintervals of equal length. A table of values of f at some points in R is given in Table 12.2.11, and a graph of f with the indicated partitions is shown in Figure 12.2.12.

Table 12.2.11 Table of values of $f(x,y) = \sqrt{4-y^2}$.

$x \downarrow \backslash y \rightarrow$	-2	-1	0	1	2
1	0	$\sqrt{3}$	2	$\sqrt{3}$	0
2	0	$\sqrt{3}$	2	$\sqrt{3}$	0
3	0	$\sqrt{3}$	2	$\sqrt{3}$	0
4	0	$\sqrt{3}$	2	$\sqrt{3}$	0
5	0	$\sqrt{3}$	2	$\sqrt{3}$	0
6	0	$\sqrt{3}$	2	$\sqrt{3}$	0
7	0	$\sqrt{3}$	2	$\sqrt{3}$	0



Standalone

In Context

Embed

Figure 12.2.12 A plot of the surface given by $f(x,y) = \sqrt{4-y^2}$

(a) Sketch the region R in the plane partitioned as described above.

(b) Calculate the double Riemann sum using the given partition of R and the values of f in the upper right corner of each subrectangle.

(c) Use geometry to calculate the exact value of $\displaystyle\iint_R f(x,y)\,dA$ and compare it to your approximation. Write a sentence to describe one way to obtain a better approximation using the given data.

Activity 12.2.4. Let $f(x,y) = x + 2y$ and let $R = [0,2] \times [1,3]$.

(a) Draw a picture of R. Partition $[0,2]$ into 2 subintervals of equal length and the interval $[1,3]$ into two subintervals of equal length. Draw these partitions on your picture of R and label the resulting subrectangles using the labeling scheme we established in the definition of a double Riemann sum.

(b) For each i and j, let (x_{ij}^*, y_{ij}^*) be the midpoint of the rectangle R_{ij}. Identify the coordinates of each (x_{ij}^*, y_{ij}^*). Draw these points on your picture of R.

(c) Calculate the Riemann sum

$$\sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*) \cdot \Delta A$$

using the partitions we have described. If we let (x_{ij}^*, y_{ij}^*) be the midpoint of the rectangle R_{ij} for each i and j, then the resulting Riemann sum is called a *midpoint sum*.

(d) Explain the meaning of the sum you just calculated in terms of each of the interpretations in Key Idea 12.2.8

12.3 Iterated Integrals

Preview Activity 12.3.1. In this activity we will explore the double integral of $f(x,y) = 25 - x^2 - y^2$ on the rectangular domain R containing the points that satisfy $-3 \leq x \leq 3$ and $-4 \leq y \leq 4$. As with partial derivatives, we may treat one of the variables in f as constant and think of the resulting function as a function of a single variable. We will now investigate what this means geometrically and algebraically when we integrate with one input variable fixed.

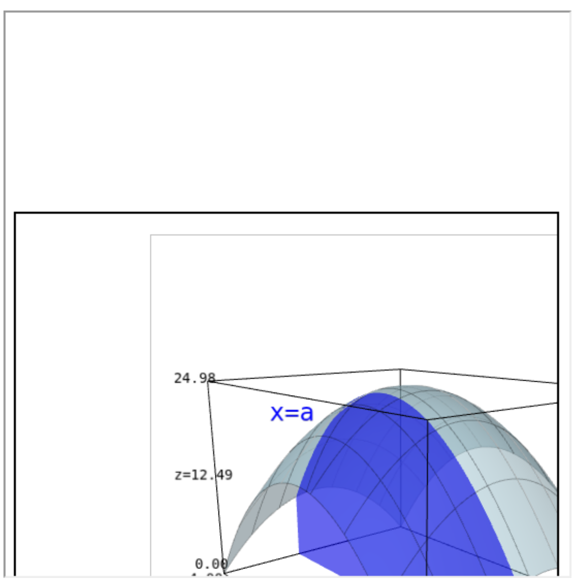
(a) Let a be a fixed value in the interval $[-3, 3]$. Compute

$$\int_{-4}^{4} f(a,y)dy.$$

(b) The answer of the previous part depended on the value of a, but the process would be the exact same set of steps for every a. This allows us to define a function $A(x)$ as

$$A(x) = \int_{-4}^{4} f(x,y)\, dy.$$

Write a couple of sentences to explain the geometric meaning of the value of $A(x)$. Your explanation should refer to different parts of Figure 12.3.2, including the surface defined by f and the trace determined by the fixed value of x.



Standalone

In Context

Embed

Figure 12.3.2 A plot of the surface $z = 25 - x^2 - y^2$ with a cross section shown at $x = a$

(c) Write a couple of sentences explaining the geometric meaning of $A(x_i^*)\ \Delta x$, where x_i^* is a fixed value of x. Your explanation should refer to the parts of Figure 12.3.3.

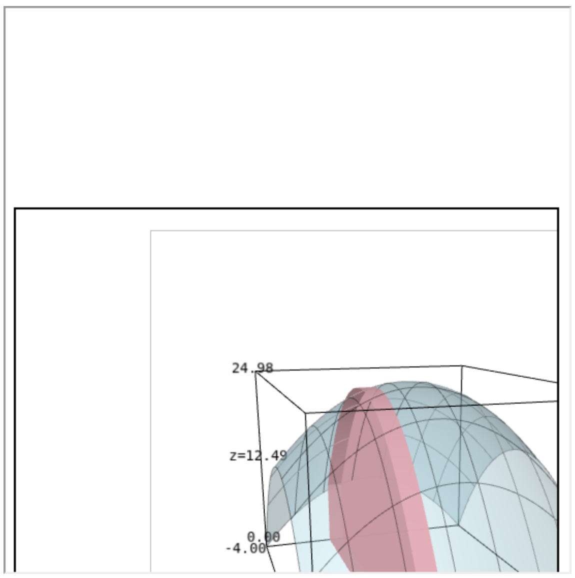


Figure 12.3.3

Standalone

In Context

Embed

(d) Since f is continuous on R, we can define the function $A = A(x)$ at every value of x in $[-3, 3]$. Think about subdividing the x-interval $[-3, 3]$ into m subintervals and choosing a value x_i^* in each of those subintervals. Write a sentence describing the meaning of the sum $\sum_{i=1}^{m} A(x_i^*)\ \Delta x$.

(e) Write a couple of sentences to explain why $\int_{-3}^{3} A(x)\,dx$ determines the exact value of the volume under the surface $z = f(x, y)$ over the rectangle R.

Activity 12.3.2. Let $f(x, y) = 25 - x^2 - y^2$ on the rectangular domain R with $-3 \le x \le 3$ and $-4 \le y \le 4$.

(a) Viewing x as a fixed constant, use the Fundamental Theorem of Calculus to evaluate the integral

$$A(x) = \int_{-4}^{4} f(x, y)\, dy.$$

Note that you will be integrating with respect to y, and holding x constant. Your result should be a function of x only.

(b) Next, use your result from (a) along with the Fundamental Theorem of Calculus to determine the value of $\int_{-3}^{3} A(x)\, dx$.

(c) What is the value of $\iint_R f(x, y)\, dA$? Write a sentence to interpret the meaning of this value for two out of the three different ways mentioned in Key Idea 12.2.8.

Activity 12.3.3. Let $f(x, y) = x + y^2$ on the rectangle R with $0 \le x \le 2$ and $0 \le y \le 3$.

(a) Evaluate $\iint_R f(x, y)\,dA$ using an iterated integral. Choose an order for integration by deciding whether you want to integrate first with respect to x or y.

(b) Evaluate $\iint_R f(x, y)\,dA$ using the iterated integral whose order of integration is the opposite of the order you chose in (a).

12.4 Double Integrals over General Regions

Preview Activity 12.4.1. A tetrahedron is a three-dimensional figure with four faces, each of which is a triangle. A picture of the tetrahedron T with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ is shown in Figure 12.4.1. By using techniques from Section 9.7, we can find that an equation for this plane is

$$z = 1 - (x + y).$$

However, we cannot directly use the methods we have developed so far involving double integrals to find the volume of a solid above the xy-plane and below a surface $z = f(x,y)$, as the base of the tetrahedron is not rectangular. This Preview Activity generalizes the work we have done so far with double integrals to understand how we can use a double integral to compute the volume of this tetrahedron. To do this, we will apply the Classic Calculus Approach.

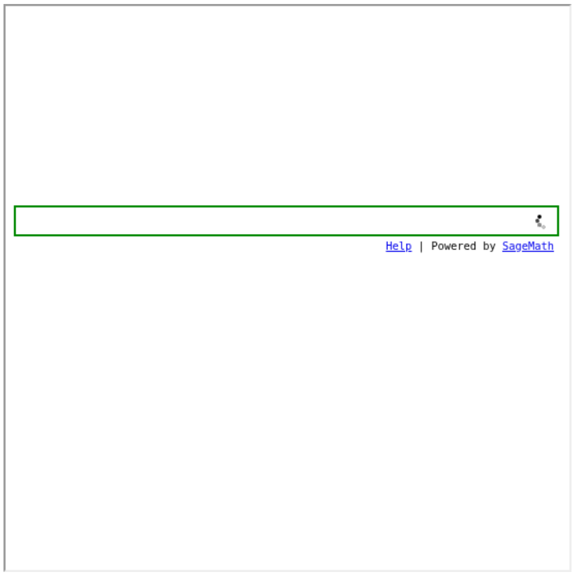
Standalone

In Context

Embed

Figure 12.4.1 The tetrahedron T with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$

(a) As our first step in applying the Classic Calculus Approach, we approximate the tetrahedron by using three triangular prisms with bases parallel to the yz-plane. Specifically, the bases lie in the planes $x = 0$, $x = 1/3$, and $x = 2/3$. We show these three triangular prisms in Figure 12.4.2.



Standalone

In Context

Embed

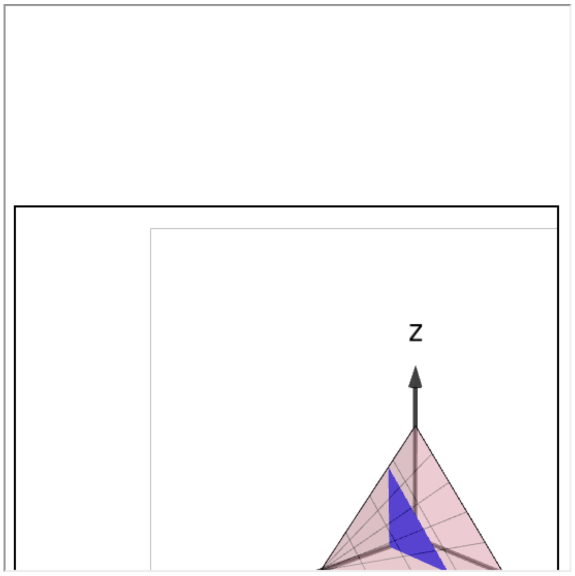
Figure 12.4.2 A decomposition of the tetrahedron into three triangular prisms with base parallel to the yz-plane

The volume of each triangular prism is the cross sectional area of the prism's base times the thickness of the prism. Find the cross sectional area for the three prisms shown in the figure.

(b) Write a sum in the form below that completes Step 1 of the Classic Calculus Approach.

$$\sum_{i=1}^{3} (\text{cross sectional area}) \cdot (\text{prism width})$$

(c) For Step 2 of the Classic Calculus Approach, we generalize the approximation above to work for n triangular prisms of equal width. To do this, we must find the cross sectional area at an arbitrary location $x = a$. Create a carefully-labeled two-dimensional drawing of the highlighted cross section in Figure 12.4.3 that can be interpreted for any value of a where the plane $x = a$ intersects the tetrahedron. Your drawing should be in the yz-plane, and you should label components of your diagram in terms of the parameter a. Make sure to write an equation for the line that forms the hypotenuse of the triangle, which should be expressed in terms of y, z, and a.



Standalone

In Context

Embed

Figure 12.4.3 The tetrahedron T with cross section along $x = a$ shown

(d) Write a dy integral for the area of the cross section you drew in the previous part. Your integral will include a in multiple ways. Verify that your integral gives the same value you found earlier for the $a = 1/3$ cross sectional area.

(e) To complete Step 2 of the Classic Calculus Approach, we approximate the volume by a sum. Specifically, if we denote the cross sectional area of the ith triangular prism by CSA_i, then the volume is approximated by

$$\sum_{i=1}^{n} \mathrm{CSA}_i \cdot \Delta x.$$

By the Classic Calculus approach, the volume of the tetrahedron is

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{CSA}_i \cdot \Delta x = \int_0^1 \mathrm{CSA(x)}\, dx, \tag{12.4.1}$$

where $\mathrm{CSA}(x)$ denotes the cross sectional area for position x.

In the previous part of the preview activity, you wrote a dy-integral for the cross sectional area at position $x = a$. Replace all occurrences of a in that integral by x and substitute the result integral into the integral in (12.4.1). This produces an iterated integral that can be used to evaluate $\iint_R 1-(x+y)\,dA$, where R is the base of the tetrahedron.

(f) Compute $\int_0^{1-x} 1-(x+y)\,dy$. Remember that to do this, you treat x as a constant, much like you would when computing a partial derivative, to find an antiderivative of $1-(x+y)$ *with respect to* y.

(g) Write your answer from the previous part in the blank in in the following integral and then evaluate the integral. Your answer will be a constant.

$$\int_0^1 \boxed{}\, dx$$

Activity 12.4.2. Let D be the region inside the unit circle centered at the origin, let R be the right half of D, and let B be the bottom half of D.

(a) On three separate plots, graph and label the regions D, R, and B.

(b) For each double integral below, decide *without calculation* whether the double integral is positive, negative, or zero. Write a sentence or two to explain your answer for each part.

i. $\displaystyle\iint_D 1\, dA$

ii. $\displaystyle\iint_D x\, dA$

iii. $\displaystyle\iint_B x\, dA$

iv. $\displaystyle\iint_R x\, dA$

v. $\displaystyle\iint_D x^2\, dA$

vi. $\displaystyle\iint_B 2 - x^2\, dA$

vii. $\displaystyle\iint_R -e^x\, dA$

viii. $\displaystyle\iint_B 1 - e^y\, dA$

ix. $\displaystyle\iint_D xy\, dA$

Activity 12.4.3. In this activity, we will break the region D from Example 12.4.12 into smaller regions which are horizontally or vertically simple.

(a) In Figure 12.4.16, the region D is broken into two regions, V_1 and V_2. Give inequalities for each of V_1 and V_2 that shows each region is vertically simple.

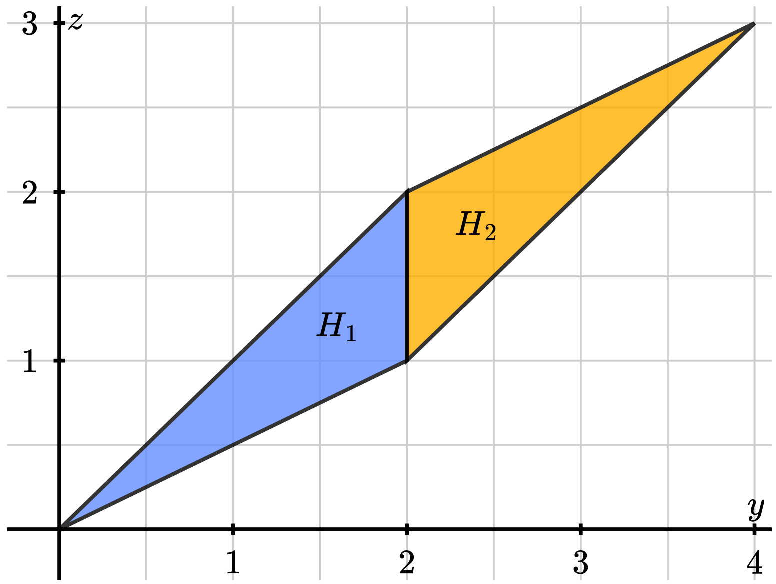


Figure 12.4.16 The region D split into two regions, V_1 and V_2, which are individually vertically simple

(b) In Figure 12.4.17, the region D is broken into three regions H_1, H_2, and H_3. Give inequalities for each of H_1, H_2, and H_3 that shows that each region is horizontally simple.

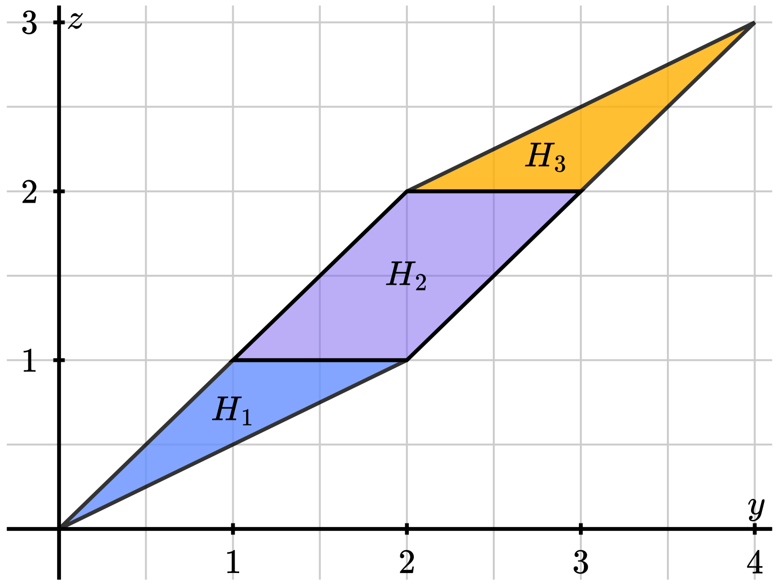
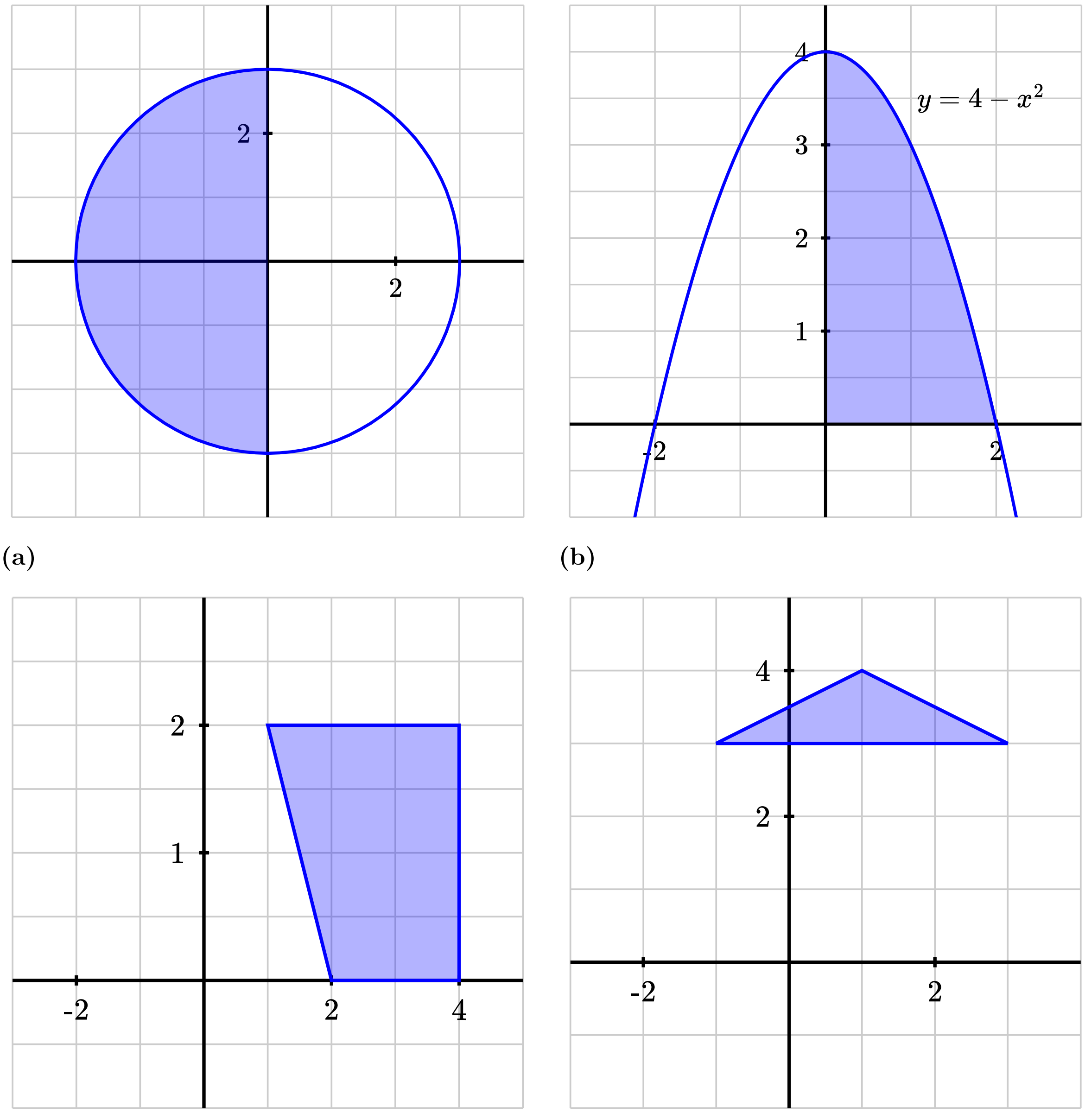


Figure 12.4.17 The region D split into three regions, H_1, H_2, and H_3, which are individually horizontally simple

Activity 12.4.4.

(a) For each region shown below, state whether the region is vertically simple, horizontally simple, both, or neither. State the appropriate inequalities to justify when a region is vertically simple or horizontally simple.



(a) **(b)** **(c)** **(d)**

Figure 12.4.21 Four Regions of the xy-plane

(b) For each of the interated integrals below, write out the bounds of the integral as a set of inequalities. Use these inequalities to decide if the bounds of the iterated integrals make sense to compute a double integral. For any iterated integral with bounds that do not make sense as a double integral, explain your reasoning in a couple of sentences. For any bounds of an iterated integral that does make sense as a double integral, sketch a plot of the associated region of integration.

(i) $\int_0^1 \int_1^x f(x,y)\,dy\,dx$

(ii) $\int_0^1 \int_1^y f(x,y)\,dy\,dx$

(iii) $\int_0^1 \int_x^y f(x,y)\,dy\,dx$

(iv) $\int_0^1 \int_x^1 f(x,y)\,dy\,dx$

Activity 12.4.5. Let D be the triangular region with vertices $(0,0)$, $(4,0)$, and $(0,2)$. Consider the double integral $\iint_D (4-x-2y)\,dA$.

(a) Draw and label a plot of D with relevant cross sections for a vertically simple description. Give the inequalities that show D is vertically simple.

(b) Write the double integral as an iterated integral of the form $\iint_D (4-x-2y)\,dy\,dx$, including correct bounds on the integral.

(c) Evaluate the iterated integrals from the previous part. Write a sentence describing at least one interpretation of the meaning of the value of the double integral.

Activity 12.4.6. Let D be the triangular region with vertices $(0,0)$, $(4,0)$, and $(0,2)$. Consider the double integral $\displaystyle\iint_D (4 - x - 2y)\, dA$.

(a) Draw and label a plot of D with relevant cross sections for a horizontally simple description. Give the inequalities that show D is horizontally simple.

(b) Write the double integral as an iterated integral of the form $\iint_D (4-x-2y)\, dx\, dy$, include correct bounds on the integral.

(c) Evaluate the iterated integral above, and verify that you get the same value as in Activity 12.4.5.

Activity 12.4.7. Consider the iterated integral $\int_{x=0}^{x=1} \int_{y=x}^{y=\sqrt{x}} (4x + 10y)\, dy\, dx$. Remember that not every iterated integral makes sense as a double integral, so as you proceed through this activity, pause to think about *why* this iterated integral does correspond to a double integral.

(a) Sketch the region of integration D for which

$$\iint_D (4x + 10y)\, dA = \int_{x=0}^{x=1} \int_{y=x}^{y=\sqrt{x}} (4x + 10y)\, dy\, dx.$$

(b) Determine the equivalent iterated integral that results from integrating in the opposite order ($dx\, dy$, instead of $dy\, dx$). That is, determine limits of integration for which

$$\iint_D (4x + 10y)\, dA = \int_{y=?}^{y=?} \int_{x=?}^{x=?} (4x + 10y)\, dx\, dy.$$

(c) Evaluate whichever one of the two iterated integrals above you find easier to compute. Explain what the value means in at least one of the interpretations of Key Idea 12.2.8.

(d) Set up and evaluate a *single* definite integral to determine the exact area $A(D)$ of D.

(e) Determine the exact average value of $f(x, y) = 4x + 10y$ over D.

12.5 Applications of Double Integrals

Preview Activity 12.5.1. Suppose that we have a flat, thin object (called a **lamina**) whose density varies across the object. We can think of the density on a lamina as a measure of mass per unit area. As an example, consider a circular plate D of radius 1 cm centered at the origin whose density δ varies depending on the distance from its center so that the density in grams per square centimeter at point (x, y) is

$$\delta(x, y) = 10 - 2(x^2 + y^2).$$

(a) Partition the plate into subrectangles R_{ij}, where $1 \le i \le m$ and $1 \le j \le n$, so that each subrectangle has area ΔA. Select a point (x_{ij}^*, y_{ij}^*) in R_{ij} for each i and j. Write a couple of sentences to explain the meaning of the quantity $\delta(x_{ij}^*, y_{ij}^*)\Delta A$.

(b) Write a double Riemann sum that computes an approximation of the mass of the plate.

(c) Explain why the double integral $\iint_D \delta(x, y)\, dA$ tells us the exact mass of the plate.

(d) Write an iterated integral which, if evaluated, would give the exact mass of the plate. Do not actually evaluate the integral. (This integral is considerably easier to evaluate in polar coordinates, which we will learn more about in Section 12.6.)

Activity 12.5.2. Suppose we want to find the area of the bounded region D between the curves

$$y = 1 - x^2 \qquad \text{and} \qquad y = x - 1$$

A picture of this region is shown in Figure 12.5.1.

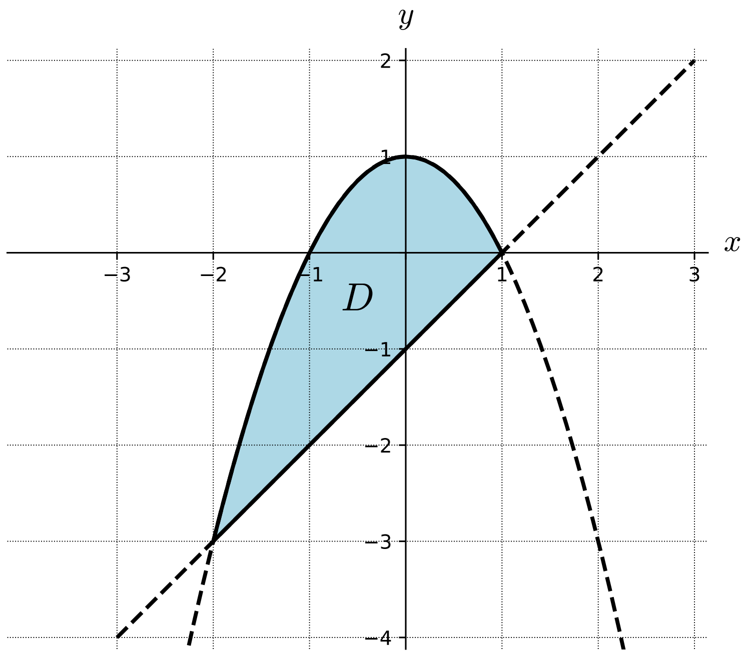


Figure 12.5.1 The graphs of $y = 1 - x^2$ and $y = x - 1$

(a) The volume of a solid with constant height is given by the area of the base times the height. Hence, we may interpret the area of the region D as the volume of a solid with base D and of uniform height 1. That is, the area of D is given by $\iint_D 1\, dA$. Write an iterated integral whose value is $\iint_D 1\, dA$.

(b) Evaluate the iterated integral from (a). What does the result tell you?

Activity 12.5.3. Let D be a half-disk lamina of radius 3 cm oriented in quadrants IV and I, centered at the origin as shown in Figure 12.5.2. Assume the density at point (x, y) is given by $\delta(x, y) = x$ with units of grams per square centimeter. Find the exact mass of the lamina.

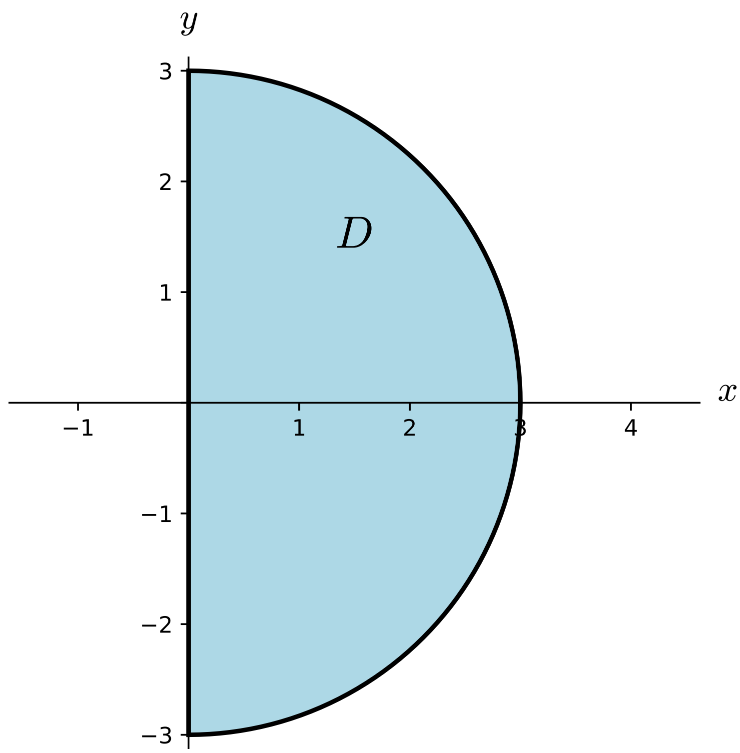


Figure 12.5.2 A half-disk lamina

Activity 12.5.4. A firm manufactures smoke detectors. Two components for the detectors come from different suppliers—one in Michigan and one in Ohio. The company studies these components for their reliability and their data suggests that if x is the life span in years of a randomly chosen component from the Michigan supplier and y the life span in years of a randomly chosen component from the Ohio supplier, then the joint probability density function f could be given by

$$f(x, y) = e^{-x}e^{-y}.$$

(a) Theoretically, the components might last forever, so the domain D of the function f is the set D of all (x, y) such that $x \geq 0$ and $y \geq 0$. To show that f is a probability density function on D we need to demonstrate that

$$\iint_D f(x, y)\, dA = 1,$$

or that

$$\int_0^\infty \int_0^\infty f(x, y)\, dy\, dx = 1.$$

Use your knowledge of improper integrals to verify that f is indeed a probability density function.

(b) Assume that the smoke detector fails only if both of the supplied components fail. To determine the probability that a randomly selected detector will fail within one year, we will need to determine the probability that the life span of each component is between 0 and 1 years. Set up an appropriate iterated integral, and evaluate the integral to determine the probability.

(c) What is the probability that a randomly chosen smoke detector will fail between years 3 and 7?

(d) Suppose that the manufacturer determines that one of the components is more likely to fail than the other, and hence conjectures that the probability density function is instead $f(x, y) = Ke^{-x}e^{-2y}$. What is the value of K?

Activity 12.5.5. In this activity we determine integrals that represent the center of mass of a lamina D described by the triangular region bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant if the density at point (x, y) is $\delta(x, y) = 6x + 6y + 6$. A picture of the lamina is shown in Figure 12.5.4.

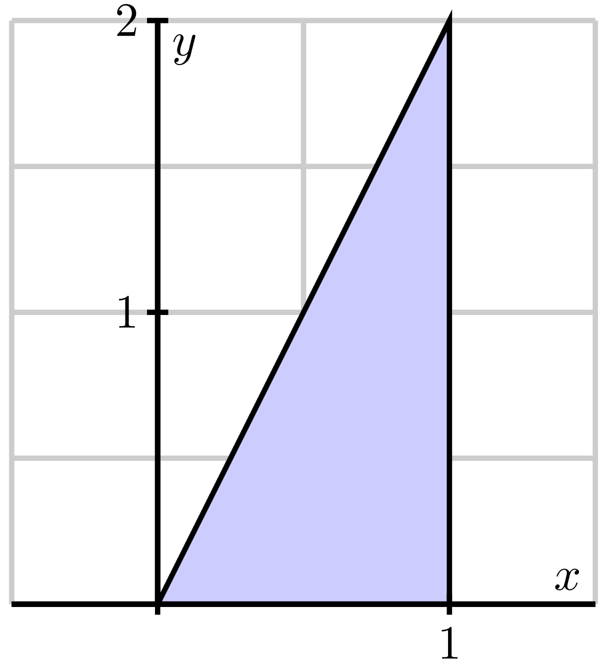


Figure 12.5.4 The lamina bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant.

(a) Set up an iterated integral that computes the total mass of the lamina.

(b) Assume the total mass of the lamina is 14. Set up two iterated integrals that represent the coordinates of the center of mass of the lamina.

12.6 Double Integrals in Polar Coordinates

Preview Activity 12.6.1. This Preview Activity asks you to practice converting some equations and regions between rectangular and polar coordinates.

(a) For each of the following polar equations, draw a plot in the plane of the curve. Write a sentence for each graph to explain why the equation is satisfied.

i. $r = 1$ ii. $r = 3$ iii. $r = 0$ iv. $\theta = 1$ v. $\theta = \frac{3\pi}{4}$

(b) Draw a plot of the region described by the following inequalities:

$$1 \leq r \leq 3 \quad \text{and} \quad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$$

(c) Sketch the region in the xy-plane with $x \leq 0$, $y \leq 0$, and inside the circle $x^2 + y^2 = 25$. State inequalities for r and θ that describe the region.

(d) Sketch the region in the xy-plane that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. State inequalities for r and θ that describe the region.

Activity 12.6.2. In this activity, we will find the volume of the solid region V above the xy-plane, below the paraboloid $z = f(x, y) = 25 - x^2 - y^2$, and outside the cylinder $x^2 + y^2 = 1$.

(a) We will find the volume by computing a double integral $\iint_R f(x, y)\, dA$ where R is an appropriate region in the xy-plane. Sketch the region R.

(b) Write inequalities to describe R in polar coordinates. Write a couple of sentences explaining why it would be very difficult to describe R using rectangular coordinates.

(c) Use an interated integral in polar coordinates to find the volume of V.

Activity 12.6.3. Use polar coordinates to find the volume of the solid between the cone $z = f(x, y) = \sqrt{x^2 + y^2}$ and the paraboloid $z = g(x, y) = \frac{1}{8}\left(x^2 + y^2\right) + 2$.

12.7 Triple Integrals

Preview Activity 12.7.1. In this activity, we want to try to estimate the mass of large piece of granite. Granite is composed of different minerals such as feldspar and quartz that give distinctive patterns of color and texture. The large piece of granite we are looking at is 4 feet wide, six feet deep, and 8 feet tall which we will describe by

$$B = \{(x, y, z) : 0 \leq x \leq 4, 0 \leq y \leq 6, 0 \leq z \leq 8\}.$$

This very special piece of granite formed in a region with many geological folds and has its density given by $\delta(x, y, z) = 163 + 3\sin(xy) + \frac{2z}{3}$. The units for x, y, z are measured in feet and the density is given in pounds per cubic foot.

For a solid of constant density, we can find the mass by multiplying the density and volume. For our block of granite, the density varies from point to point based on the function $\delta(x, y, z)$. In this activity, we will approximate the mass of the block (step 1 of the classic calculus approach) by slicing the solid into smaller pieces on which there are smaller density changes and thus the density is closer to constant.

(a) For a first approximation using smaller pieces, we partition the block as follows

- the width, given by the x-interval $[0, 4]$, into two subintervals of equal length
- the depth, given by the y-interval $[0, 6]$ into three subintervals of equal length
- the height, given by the z-interval $[0, 8]$ into two subintervals of equal length

This partitions the box B into sub-boxes as shown in Figure 12.7.1.

Standalone

In Context

Embed

Figure 12.7.1 A partitioned three-dimensional domain

Let $0 = x_0 < x_1 < x_2 = 4$ be the endpoints of the x-subintervals of $[0, 4]$ after partitioning. Label these endpoints on Figure 12.7.2. Repeat this process with $0 = y_0 < y_1 < y_2 < y_3 = 6$ and $0 = z_0 < z_1 < z_2 = 8$.

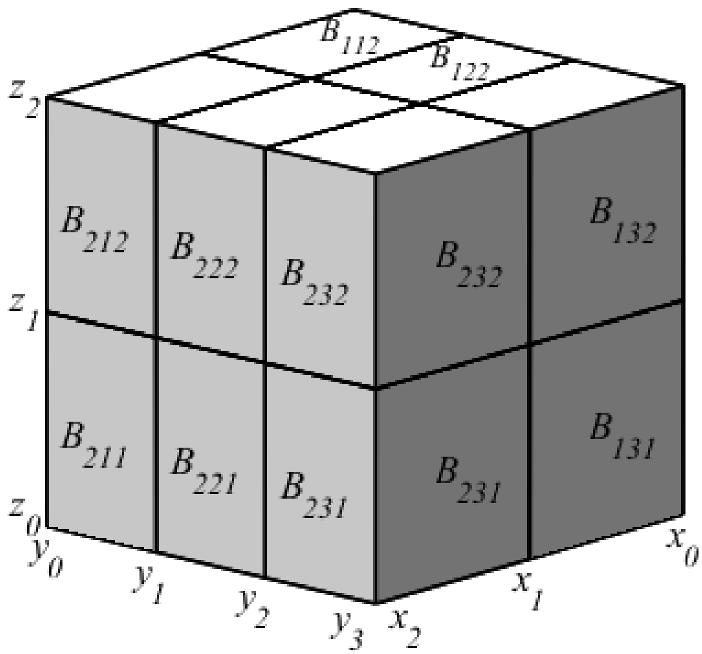


Figure 12.7.2 A partitioning of our granite block

(b) Our partitions of the x, y, z-intervals created sub-boxes for our granite block. How many sub-boxes were created and what is the volume of the sub-boxes?

(c) For each sub-box, pick a point in that region and evaluate the density function at that point. Make sure you note what point you chose and the density at that point.

(d) For each sub-box, multiply the density at your chosen point by the volume of the sub-box to get an estimate of the mass for the sub-box. Sum over all of your sub-boxes to give an estimate of the mass for the entire block, which will complete step 1 of the classic calculus approach to measuring the total mass of our granite block.

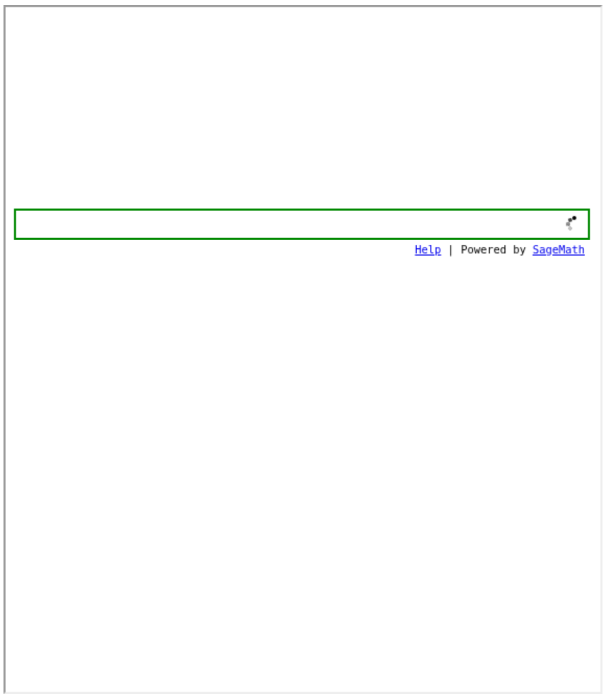
(e) We partitioned our block, the space of inputs for $\delta(x, y, z)$, into a relatively small number of pieces. Write a couple of sentences about what you would change in the process above to improve the approximation of the mass of the block. Be sure to explain why your changes will give a better approximation.

Activity 12.7.2. Set up and evaluate an iterated integral that will evaluate the triple integral of $f(x, y, z) = x - y + 2z$ over the box $B = [-2, 3] \times [1, 4] \times [0, 2]$.

Activity 12.7.3. Let S be the solid cone bounded by $z = \sqrt{x^2 + y^2}$ and $z = 3$. The goal of this activity is to set up an iterated integral of the form

$$\int_{x=?}^{x=?} \int_{y=?}^{y=?} \int_{z=?}^{z=?} \delta(x, y, z)\, dz\, dy\, dx \tag{12.7.2}$$

to represent the mass of S in the setting where $\delta(x, y, z)$ tells us the density of S at the point (x, y, z). In particular, we must find the limits on each of the three integrals.
A picture of S is shown in Figure 12.7.8. Adjust the sliders, which will move the vertical line segment shown inside the solid. Notice that some combinations of slider values show a red point and no line segment, as the point (x, y) does not correspond to any points inside the solid.



Standalone

In Context

Embed

Figure 12.7.8 The region S with a vertical segment highlighted for fixed x and y values

(a) For the innermost integral of equation (12.7.2), we need bounds on the z-coordinate for fixed values of x and y. In Figure 12.7.8, you can use the sliders to change the values of x and y. When your choices of x and y correspond to points inside the solid, you see a vertical line segment in the plot from the bottom of the solid to the top of the solid over the point (x, y) in the xy-plane.

Try several values of x and y. Look at how the length of the segment changes in the z-direction. In particular, *for every x and y pair*, the bottom boundary of the solid is the same. Similarly, for every pair of values the top boundary of the solid is the same surface. This allows you to use functions, in terms of x and y, that describe the top and bottom boundaries of the solid. These are the z-coordinates of the points at the top and bottom of the vertical line segments through the solid shown in the figure.

$$z_{\text{top}}(x, y) = \underline{\hspace{4cm}}$$
$$z_{\text{bottom}}(x, y) = \underline{\hspace{4cm}}$$

(b) Complete the compound inequality below to provide lower and upper bounds for the z-coordinates of points in S (in terms of x and y).

$$\rule{4cm}{0.4pt} \leq z \leq \rule{4cm}{0.4pt}$$

(c) Having established upper and lower bounds for z as a function of a fixed choice of x and y, we need to describe the set of points (x, y) in the xy-plane such that a vertical line through (x, y) will pass through the solid. Notice that if you choose values of x and y in Figure 12.7.8 that does not intersect the solid (e.g., $(x, y) = (2.6, -2.3)$), then the point is shown in red.

You can see from Figure 12.7.8 that there will be x and y values from -3 to 3 that will correspond to points in our solid. It is tempting to give the region of the xy-plane we need to consider using the inequalities $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$. Write a couple of sentences to explain why the set of (x, y) points we need to consider is *not* the square $[-3, 3] \times [-3, 3]$.

(d) On Figure 12.7.9, draw a plot of D, the region of (x, y) points that correspond to points of S. We refer to D as the **projection** of S onto the xy-plane.

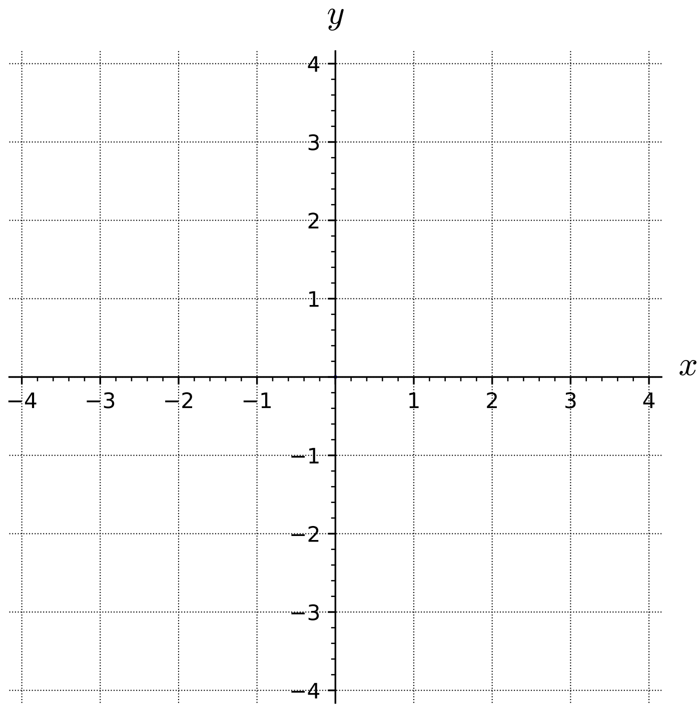


Figure 12.7.9 A grid of $[-4,4] \times [-4,4]$

(e) To complete the task of writing

$$\iiint_S \delta(x,y,z)dV$$

as an iterated integral, you need to describe the region D from the previous part using inequalities, as with double integrals. Do this using a *vertically simple* description in order to have your iterated integral fit the form of equation (12.7.2).

Give the inequalities that describe the region D from the previous part as vertically simple.

$$\underline{\hspace{4cm}} \leq x \leq \underline{\hspace{4cm}}$$
$$\underline{\hspace{4cm}} \leq y \leq \underline{\hspace{4cm}}$$

(f) Write an iterated integral of the form (12.7.2) that represents the mass of the cone S.

Activity 12.7.4. There are several other ways to set up the integral to give the mass of the tetrahedron in Example 12.7.5.

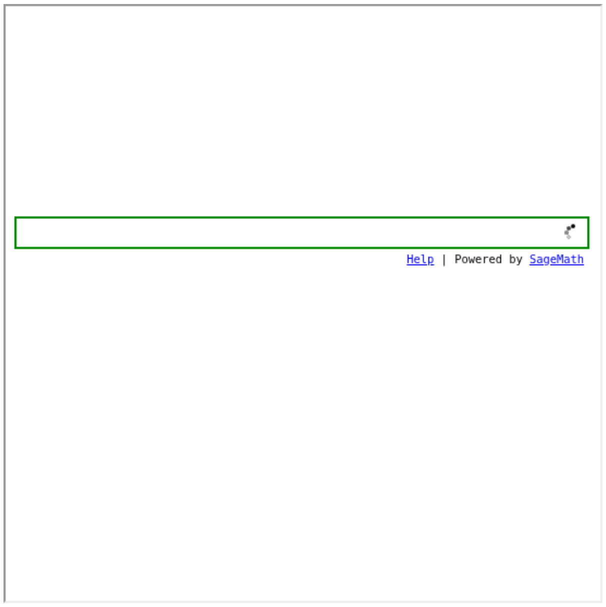
(a) How many different orders of integration could be used for iterated integrals that are equal to the integral in equation (12.7.1)?

(b) Set up an iterated integral, integrating first with respect to z, then x, then y that is equivalent to the integral in equation (12.7.1). Before you write down the integral, think about Figure 12.7.6 and draw a plot of the appropriate projection.

(c) Set up an iterated integral, integrating first with respect to y, then z, then x, that is equivalent to the integral in equation (12.7.1). As above, think carefully about the geometry first and draw a plot of the appropriate projection.

(d) Set up an iterated integral, integrating first with respect to x, then y, then z that is equivalent to the integral in equation (12.7.1).

Activity 12.7.5. A solid S is bounded below by the paraboloid $z = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 6$. A picture of S is shown in Figure 12.7.11.



Standalone

In Context

Embed

Figure 12.7.11 The solid bounded below by the paraboloid $z = x^2+y^2$ and above by the sphere $x^2+y^2+z^2 = 6$

(a) Set up an iterated triple integral to find the volume of S. You do not need to evaluate the integral at this time.

(b) Suppose the density at point (x, y, z) is $\delta(x, y, z) = x^2 + 1$. Set up, but do not evaluate, the necessary iterated integrals to find the center of mass of S.

(c) Let $f(x, y, z) = xy + z^2$. Set up but do not evaluate an iterated triple integral to find the average value of f on S.

(d) Use technology to evaluate the iterated triple integrals you wrote in the other three parts of this activity. Write a couple of sentences to explain why the location of the center of mass makes sense.

12.8 Triple Integrals in Cylindrical and Spherical Coordinates

Preview Activity 12.8.1. In this Preview Activity, we will be reviewing the meaning of each of the cylindrical and spherical coordinates by looking at a description of a common surface in either cylindrical or spherical coordinates. For each task, you should draw a plot of the surface described by hand and write a few sentences describing how your plot relates to the cylindrical/spherical coordinates.

(a) What familiar surface is described by the points in cylindrical coordinates with $r = 2$, $0 \leq \theta \leq 2\pi$, and $0 \leq z \leq 2$? How does this example suggest that we call these coordinates *cylindrical coordinates*? How does the surface change if we restrict θ to $0 \leq \theta \leq \pi$?

(b) What familiar surface is described by the points in cylindrical coordinates with $\theta = 2$, $0 \leq r \leq 2$, and $0 \leq z \leq 2$?

(c) Plot the graph of the cylindrical equation $z = r$, where $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. What familiar surface is this a plot of?

(d) What familiar surface is described by the points in spherical coordinates with $\rho = 1$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$? How does this particular example demonstrate the reason for the name of this coordinate system? What if we restrict ϕ to $0 \leq \phi \leq \frac{\pi}{2}$?

(e) What familiar surface is described by the points in spherical coordinates with $\phi = \frac{\pi}{3}$, $0 \le \rho \le 1$, and $0 \le \theta \le 2\pi$?

Activity 12.8.2. In this activity, we will work with a few triple integrals and look at the advantages of utilizing cylindrical coordinates for the iterated integral calculation.

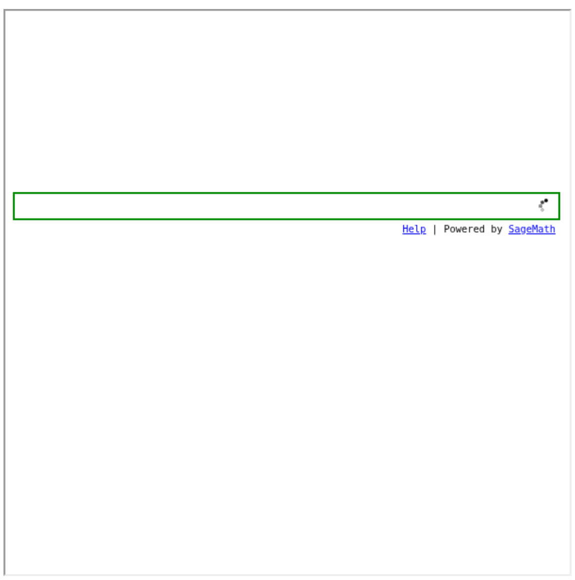
(a) Let S be the solid bounded above by the graph of $z = x^2 + y^2$ and below by $z = 0$ on the unit disk in the xy-plane.

i. The projection of the solid S onto the xy-plane is a disk. Describe this disk using inequalities in polar coordinates.

ii. Use your inequalities from above to write inequalities in cylindrical coordinates that describe the solid S.

iii. Write out a set of iterated integrals that will evaluate to the volume of S. You do not need to evaluate this integral.

(b) Suppose the density of the cone defined by $r = 1 - z$, with $z \geq 0$, is given by $\delta(r, \theta, z) = z$. A picture of the cone is shown in Figure 12.8.2. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone. You do not need to evaluate this integral.



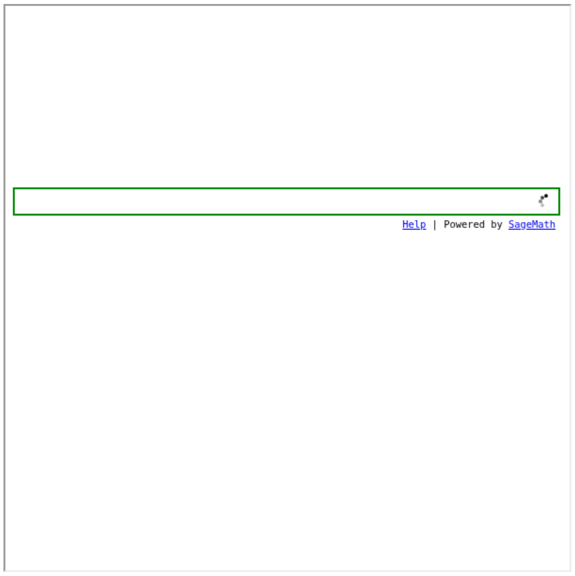
Standalone

In Context

Embed

Figure 12.8.2 The cylindrical cone $r = 1 - z$

(c) Write an iterated integral expression in cylindrical coordinates whose value is the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the cone $z = 4 - \sqrt{x^2 + y^2}$. A picture is shown in Figure 12.8.3. You do not need to evaluate this integral.



Standalone

In Context

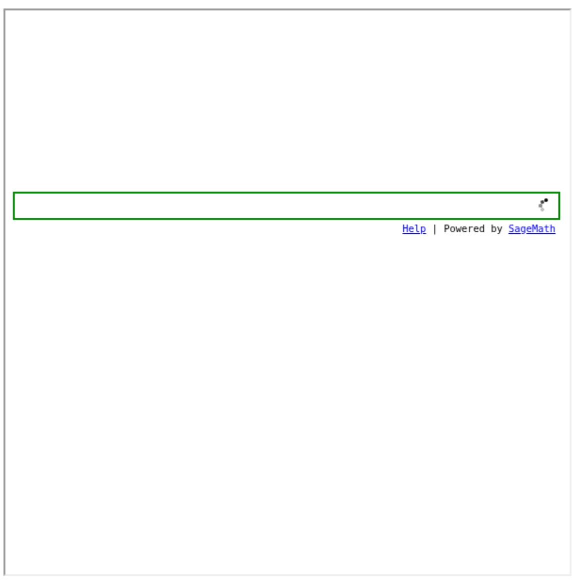
Embed

Figure 12.8.3

Activity 12.8.3. In this activity, we will use spherical coordinates to help us more easily understand some natural geometric objects.

(a) Recall that the sphere of radius a has spherical equation $\rho = a$. Set up and evaluate an iterated integral in spherical coordinates to determine the volume inside a sphere of radius a.

(b) Set up, but do not evaluate, an iterated integral expression in spherical coordinates whose value is the mass of the solid obtained by removing the region inside the cone $\phi = \frac{\pi}{4}$ from the sphere $\rho = 2$. Let δ at the point (x, y, z) be the density given by $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. An illustration of this solid is shown in Figure 12.8.7.



Standalone

In Context

Embed

Figure 12.8.7 The solid cut from the sphere $\rho = 2$ by the cone $\phi = \frac{\pi}{4}$

12.9 Change of Variables

Preview Activity 12.9.1.

(a) Consider the double integral

$$I = \iint_D x^2 + y^2 \, dA \tag{12.9.1}$$

where D is the upper half of the unit disk ($y \geq 0$). Remember the polar conversion equations:

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta) \tag{12.9.2}$$

(i) Write the double integral I given in Equation (12.9.1) as an iterated integral in rectangular coordinates.

(ii) Write the double integral I given in Equation (12.9.1) as an iterated integral in polar coordinates.

(iii) Draw a plot of the region of integration D in the xy-plane and label the four "corners" of the polar coordinates (if D is described by $a \leq r \leq b, c \leq \theta \leq d$, then the corners will be $(r, \theta) = (a, c), (a, d), (b, c), (b, d)$).

(b) Suppose you are calibrating a new radar system for your self driving car. This radar system is mounted at the center of your front bumper and can detect objects up to 45 degrees to either side of the forward direction at distances from 1 meter to 3 meters.

(i) On Figure 12.9.1, draw a plot of S, the region that your radar system can detect objects. Assume your car's front bumper is centered at the origin of your coordinate system and the car is facing the positive y-direction. You should label the four corners of S in terms of *both* rectangular and polar coordinates.

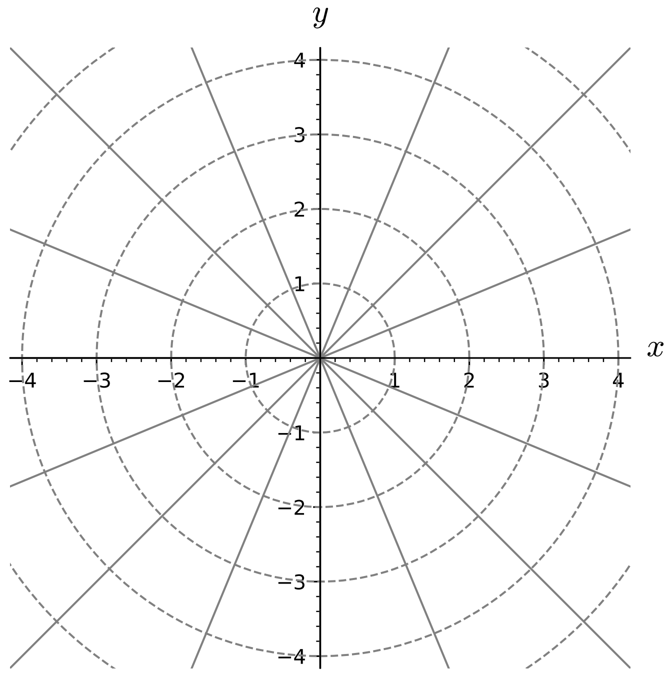


Figure 12.9.1 A polar grid

(ii) On Figure 12.9.2, plot the rectangle on the $r\theta$-plane that corresponds to the bounds $a \le r \le b \quad c \le \theta \le d$ for your radar system. Label the four corners of your figure with their r and θ coordinates.

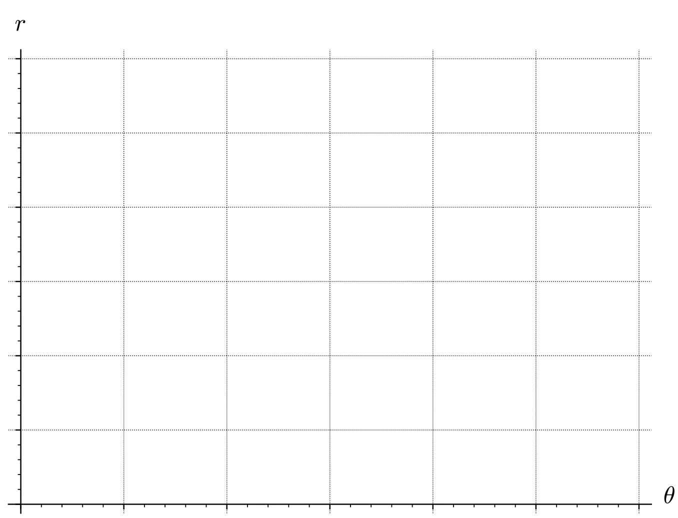


Figure 12.9.2 A grid on the θr-plane

Activity 12.9.2. In this activity, we will be looking at what happens to the region D, as shown in Figure 12.9.5, under the change of coordinates given by

$$x = \sqrt{st} \quad \text{and} \quad y = \sqrt{\frac{s}{t}}$$

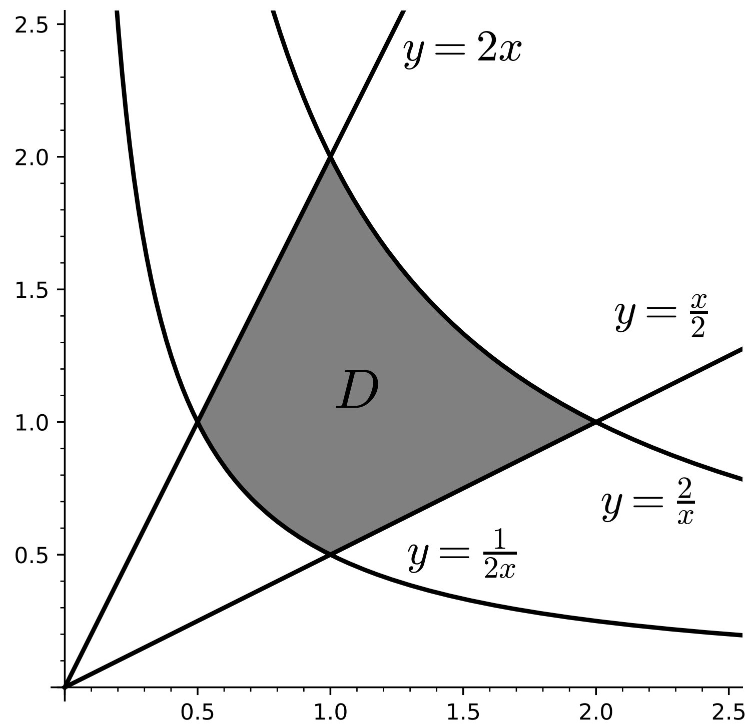


Figure 12.9.5

(a) Find the coordinates of the four corner points of the region D.

(b) Given the form of the change of variable equations above, it is not obvious how to change (x, y) points into (s, t) points. Show that the following equations are satisfied by our change of variable equations:

$$yx = s \quad \text{and} \quad \frac{y}{x} = \frac{1}{t}$$

Use these new relationships between our coordinates to convert each of these four corner points from xy-coordinates to st-coordinates and draw these points on Figure 12.9.6.

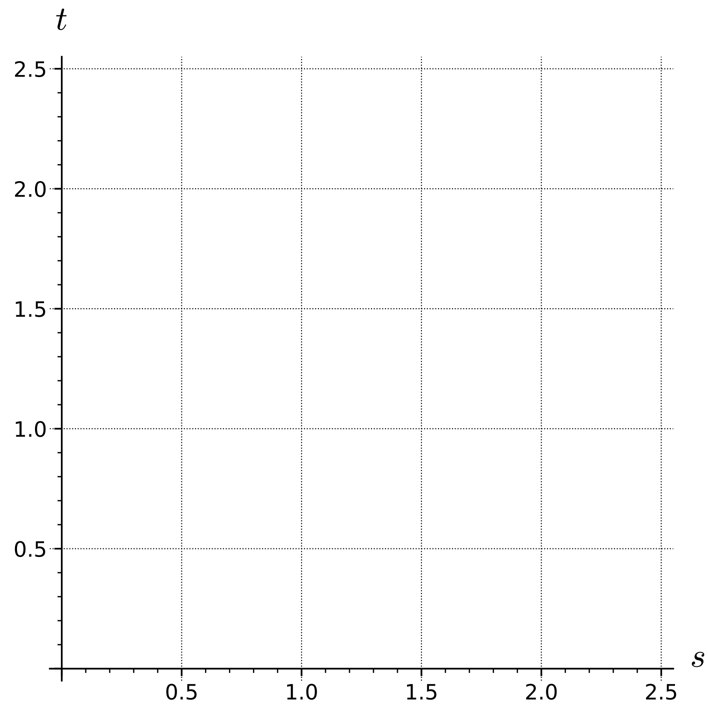


Figure 12.9.6

(c) We hope that the curved boundaries of the region D get transformed to horizontal and vertical lines in the st-plane, as is suggested by the result of the previous task. Let's look at how each of the boundary curves gets transformed by our change of variables formula.

Use the change of variables equations to write $y = 2x$ in terms of s and t coordinates and solve for either s or t (as would make sense). Plot the resulting line on Figure 12.9.6.

(d) Write each of the other boundary curves of D ($y = \frac{2}{x}, y = \frac{1}{2x}, y = \frac{x}{2}$) in terms of s and t coordinates, solve for either s or t (as would make sense), and plot the resulting lines on Figure 12.9.6.

(e) We have now confirmed that our irregular region D gets mapped to a rectangle in the st-plane by the change of variables formula given. Some areas of the region D needed to be stretched more than others in order to straighten out our sides into the nice rectangle you have drawn on Figure 12.9.6. Write a few sentences to describe what part of D you think is stretched the most in the transformation from xy-coordinates to st-coordinates.

Activity 12.9.3. In this activity, we will look at a few familiar Jacobians and their interpretations.

(a) Find the Jacobian for the transformation given by $x = 3s - 1$ and $y = \frac{t}{3} + 2$. Write a couple of sentences about why the area of a region will not change with respect to this change of coordinates.

(b) Find the Jacobian when changing from rectangular to polar coordinates. Write a couple of sentences to connect to our work on double integrals in polar coordinates.

Activity 12.9.4. Consider the problem of finding the area of the region D' defined by the ellipse $x^2 + \frac{y^2}{4} = 1$. In this activity, we will look at what happens when we use a change of variables so that the pre-image of the domain is a circle.

(a) Let $x(s,t) = s$ and $y(s,t) = 2t$. Explain why the pre-image of the original ellipse (which lies in the xy plane) is the circle $s^2 + t^2 = 1$ in the st-plane.

(b) Recall that the area of the ellipse D' is determined by the double integral $\iint_{D'} 1\,dA$. Explain why

$$\iint_{D'} 1\,dA = \iint_{D} 2\,ds\,dt$$

where D is the disk bounded by the circle $s^2 + t^2 = 1$. In particular, explain the source of the "2" in the st integral.

(c) Without evaluating any of the integrals present, explain why the area of the original elliptical region D' is 2π.

Activity 12.9.5. Let D' be the region in the xy-plane bounded by the lines $y = 0$, $x = 0$, and $x + y = 1$. We will evaluate the double integral

$$\iint_{D'} \sqrt{x+y}(x-y)^2 \, dA \tag{12.9.4}$$

with a change of variables.

(a) Sketch the region D' in the xy-plane.

(b) We would like to make a substitution that makes the integrand easier to antidifferentiate. Let $s = x + y$ and $t = x - y$. Explain why this should make antidifferentiation easier by making the corresponding substitutions and writing the new integrand in terms of s and t.

(c) Solve the equations $s = x + y$ and $t = x - y$ for x and y. (Doing so determines the standard form of the transformation, since we will have x as a function of s and t, and y as a function of s and t.)

(d) To actually execute this change of variables, we need to know the st-region D that corresponds to the xy-region D'.

i. What st equation corresponds to the xy equation $x + y = 1$?

ii. What st equation corresponds to the xy equation $x = 0$?

iii. What st equation corresponds to the xy equation $y = 0$?

iv. Sketch the st region D that corresponds to the xy domain D'.

(e) Make the change of variables indicated by $s = x + y$ and $t = x - y$ in the double integral (12.9.4) and set up an iterated integral in st variables whose value is the original given double integral. Finally, evaluate the iterated integral.

Activity 12.9.6. Consider the solid S' defined by the inequalities $0 \leq x \leq 2$, $\frac{x}{2} \leq y \leq \frac{x}{2} + 1$, and $0 \leq z \leq 6$. Consider the transformation defined by $s = \frac{x}{2}$, $t = \frac{x-2y}{2}$, and $u = \frac{z}{3}$. Let $f(x, y, x) = x - 2y + z$.

(a) The transformation turns the solid S' in xyz-coordinates into a box S in stu-coordinates. Apply the transformation to the boundries of the solid S' to find stu-coordinate descriptions of the box S.

(b) Compute and simplify the Jacobian $\frac{\partial(x,y,z)}{\partial(s,t,u)}$.

(c) Use the transformation to perform a change of variables and evaluate $\iiint_{S'} f(x, y, z)\, dV$ by evaluating

$$\iiint_S f(x(s,t,u), y(s,t,u), z(s,t,u)) \left| \frac{\partial(x,y,z)}{\partial(s,t,u)} \right| ds\, dt\, du$$

13 Vector Calculus

13.2 Vector Fields

Preview Activity 13.2.1. It's common when discussing weather to talk about the wind **speed**, but as any student who has gotten this far in the text will know, this nomenclature is imprecise. It's not terribly helpful to tell someone the wind is blowing at 10 $\frac{\text{km}}{\text{h}}$ without telling them the direction in which the wind is blowing. If you're trying to make a decision based on what the wind is doing, you need to know about the direction as well. For instance, if you are taking off in a hot air balloon, the wind direction will determine which direction the chase team should start going to keep track of you. Because of the swirling nature of wind, it makes sense to give the wind **velocity** at each point in a region (two-dimensional or three-dimensional).

(a) Suppose that given a point (x, y) in the plane, you know that the wind velocity at that point is given by the vector $\langle y, x\rangle$. For example, we'd then know that at the point $(1, -1)$, the wind velocity is $\langle -1, 1\rangle$. We will give the wind velocity as a function $\vec{F}$, where $\vec{F}(x, y) = \langle y, x\rangle$. In the table below, fill in the wind velocity vectors for the given points.

(x, y)	$(2, 1)$	$(0, 0)$	$(-1, 2)$	$(3, -1)$	$(-2, -1)$
$\vec{F}(x, y)$					

(b) Suppose that we associate the vector $\vec{G}(x, y) = -x\hat{\jmath}$ to a point (x, y) in the plane. Complete the table below by giving the vector associated to each of the given points.

(x, y)	$(-2, 0)$	$(-1, 2)$	$(0, -2)$	$(2, 3)$	$(3, 2)$	$(-1, 0)$	$(1, 3)$
$\vec{G}(x, y)$							

(c) A table of values of these vector-valued functions is useful to understand the input vs. output nature of a vector field as a function, but perhaps even better is a method of visualizing the vector outputs. A good picture is worth a thousand words (or numbers). Returning to our analogy of the output vector for our vector field being wind velocity, if $\vec{F}(2,1) = \langle 1, 2 \rangle$, this means that at the location $(2,1)$ the wind is moving in the direction given by $\langle 1, 2 \rangle$. Thus, we draw the output vector $\langle 1, 2 \rangle$ with its initial point at $(2,1)$.

Using the first set of axes in Figure 13.2.1, plot the vectors $\vec{F}(x,y)$ for the five points in the table in part a. The example $\vec{F}(1,-1) = \langle -1, 1 \rangle$ is drawn for you.

(d) Using the second set of axes in Figure 13.2.1, plot the vectors $\vec{G}(x,y)$ for the eight points in the table in part b.

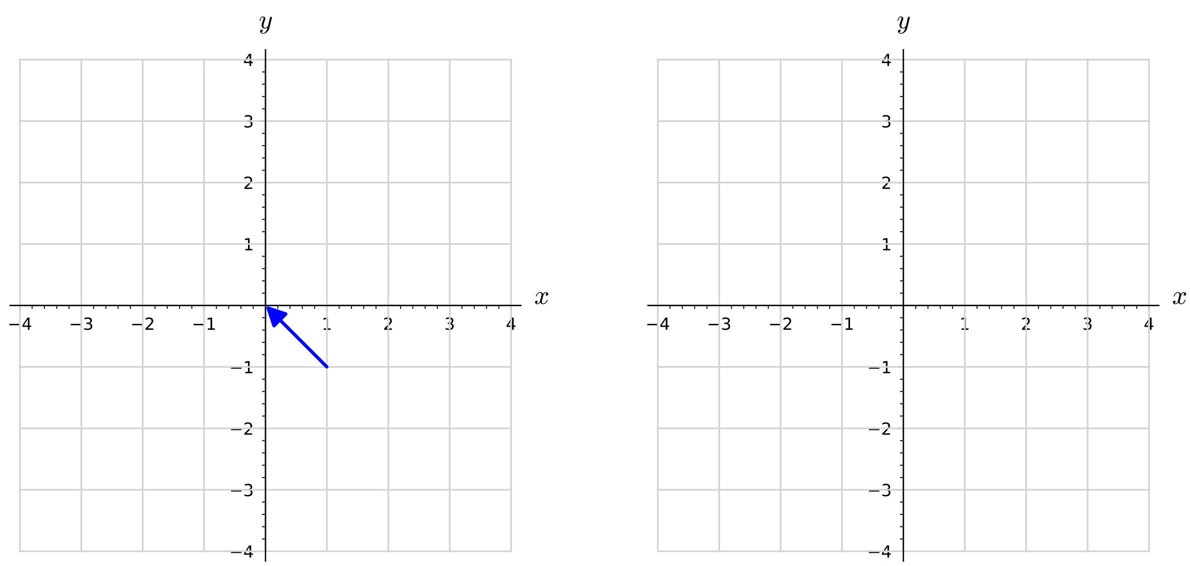


Figure 13.2.1 Axes for plotting some vectors from $\vec{F}(x,y)$ and $\vec{G}(x,y)$.

Activity 13.2.2. The plot in Figure 13.2.6 illustrates the vector field $\vec{F}(x,y) = y\hat{\imath} - x\hat{\jmath}$.

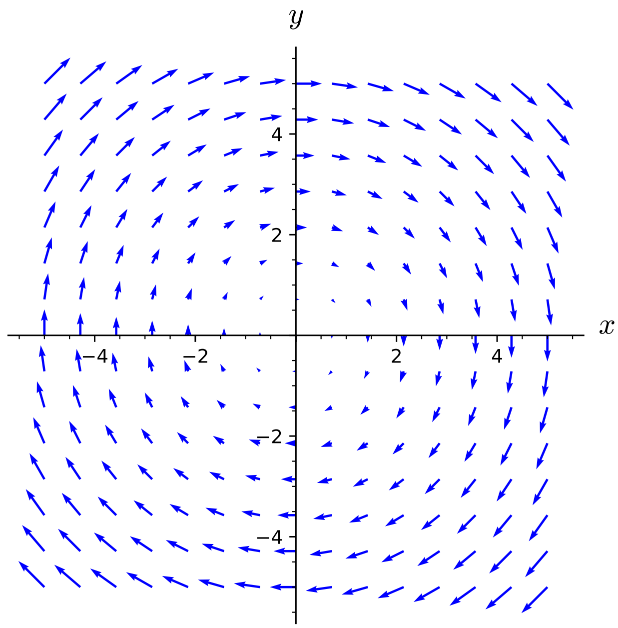


Figure 13.2.6 The vector field $y\hat{\imath} - x\hat{\jmath}$

(a) Starting with one of the vectors near the point $(2,0)$, sketch a curve that follows the direction of the vector field $\vec{F}$. To help visualize what you are doing, it may be useful to think of the vector field as the velocity vector field for some flowing water and that you are imagining tracing the path that a tiny particle inserted into the water would follow as the water moves it around.

(b) Repeat the previous step for at least two other starting points not on the curve you previously sketched.

(c) What shape do the curves you sketched in the previous two steps form?

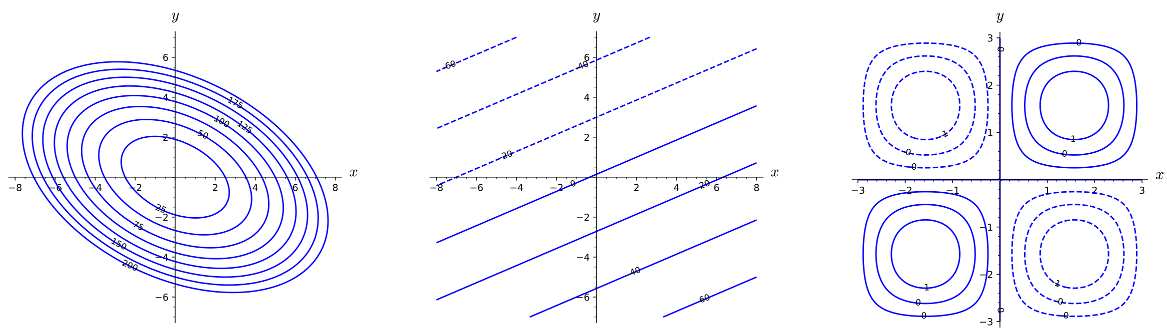
(d) Verify that $\vec{F}(x, y)$ is orthogonal to $\langle x, y \rangle$.

(e) Calculate the gradient of the function $f(x, y) = x^2 + y^2$ and write a sentence comparing your result to the vector $x\hat{\imath} + y\hat{\jmath}$.

(f) Write a sentence describing the geometric relationship between $\vec{F}(x, y)$ and a circle centered at the origin. What is the relationship between $\left\| \vec{F}(x, y) \right\|$ and the radius of that circle?

Activity 13.2.3.

(a) In Figure 13.2.7 there are three sets of axes showing level curves for functions f, g, and h, respectively. Sketch at least six vectors in the gradient vector field for each function. In making your sketches, you don't have to worry about getting vector magnitudes precise, but you should ensure that the relative magnitudes (and directions) are correct for each function independently.



(a) f **(b)** g **(c)** h

Figure 13.2.7 Three sets of level curves

(b) Verify that $\vec{F}(x,y) = \langle 6xy, 3x^2 + 9\sqrt{y}\rangle$ is a gradient vector field by finding a function f such that $\nabla f(x,y) = \vec{F}(x,y)$. For reasons originating in physics, such a function f is called a **potential function** for the vector field $\vec{F}$.

(c) Is the function f found in part b unique? That is, can you find another function g such that $\nabla g(x,y) = \vec{F}(x,y)$ but $f \neq g$?

(d) Is the vector field $\vec{F}(x,y) = 6xy\hat{\imath} + (2x + 9\sqrt{y})\hat{\jmath}$ a gradient vector field? Why or why not?

13.3 The Idea of a Line Integral

Preview Activity 13.3.1. Recall from Section 9.4 that the work done by a force $\vec{F}$ on an object that moves with displacement vector $\vec{v}$ is $\vec{F}\cdot\vec{v}$. In this Preview Activity, we consider the work done by wind on a helicopter at various stages of its journey.

(a) Our intrepid pilot flies for some time and finds that they are 30 km from where they started at a heading of 20° degrees east of due north. During this portion of the trip, the wind is exerting a force of 100 N on the helicopter in the due east direction. Find the work the wind has done on the helicopter during the flight.

(b) Our pilot sees a storm ahead and changes their direction. Some time later, the pilot determines that they are 25 km due north of where they previously checked their position. The wind is still exerting a force on the helicopter of 100 N in the due east direction. Find the work done by the wind on the helicopter during the second part of the flight.

(c) Find the helicopters's displacement from its original position after the first two parts of its flight and use that to find the work done by the wind on the helicopter during the first two parts of flight.

(d) How does your answer to part 13.3.1.c connect to the answers to part 13.3.1.a and part 13.3.1.b?

(e) In order to get further away from the storm, the pilot turns and flies 45° west of due north for 50 km. The storm the pilot was avoiding has caused the wind to change as well. For this portion of the flight, the wind is exerting a force on the helicopter of 125 N in the south direction. Find the work done by the wind on the helicopter during this part of the flight.

(f) Explain why you cannot take the total displacement of the three parts of the helicopter flight and calculate the total work done by the wind on the helicopter.

Activity 13.3.2. For each curve below, find a parametrization of the curve. Ensure that each curve's orientation matches the one specified.

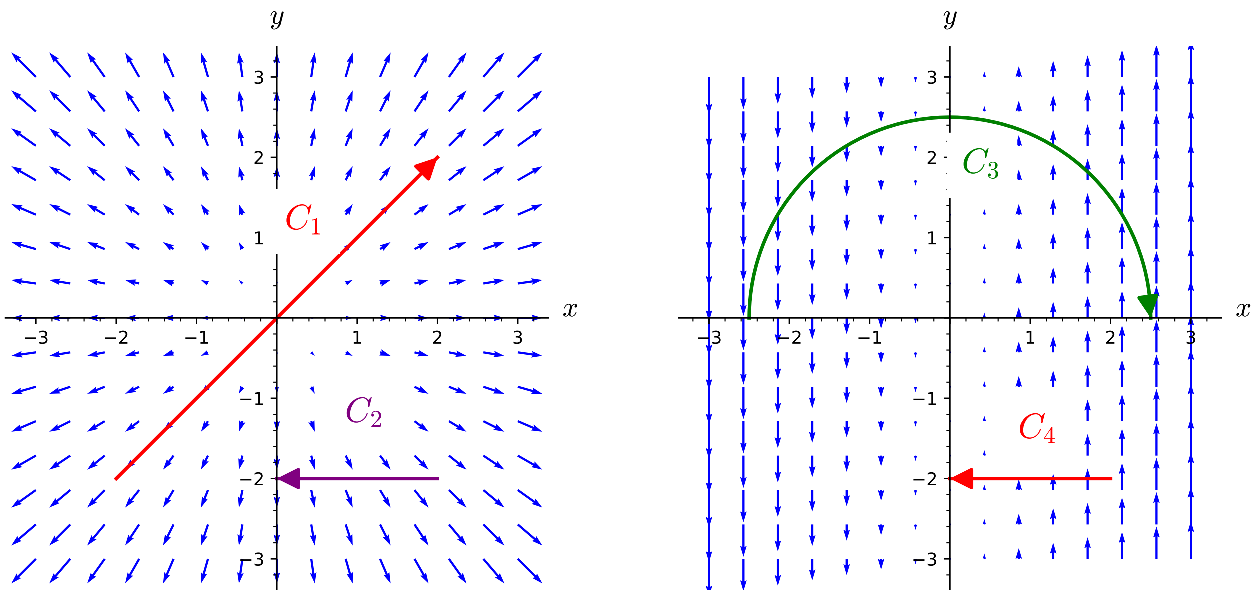
(a) The line segment in $\mathbb{R}^3$ from $(0, 1, -2)$ to $(3, -1, 2)$.

(b) The line segment in $\mathbb{R}^3$ from $(3, -1, 2)$ to $(0, 1, -2)$.

(c) The circle of radius 3 (in $\mathbb{R}^2$) centered at the origin, beginning at the point $(0, -3)$ and proceeding clockwise around the circle.

(d) In $\mathbb{R}^2$, the portion of the parabola $y^2 = x$ from the point $(4, 2)$ to the point $(1, -1)$.

Activity 13.3.3. Shown in Figure 13.3.10 are two vector fields, $\vec{F}$ and $\vec{G}$ and four oriented curves, as labeled in the plots. For each of the line integrals below, determine if its value should be positive, negative, or zero. Do this by thinking about if the vector field is helping or hindering a particle moving along the oriented curve, rather than by doing calculations.



(a) A plot of $\vec{F}$ with paths C_1 and C_2

(b) A plot of $\vec{G}$ with paths C_3 and C_4

Figure 13.3.10 Vector fields and oriented curves

(a) $\displaystyle\int_{C_1} \vec{F} \cdot d\vec{r}$

(b) $\displaystyle\int_{C_2} \vec{F} \cdot d\vec{r}$

(c) $\displaystyle\int_{C_3} \vec{G} \cdot d\vec{r}$

(d) $\displaystyle\int_{C_4} \vec{G} \cdot d\vec{r}$

Activity 13.3.4. Figure 13.3.15 shows a vector field $\vec{F}$ as well as six oriented curves, as labeled in the plot.

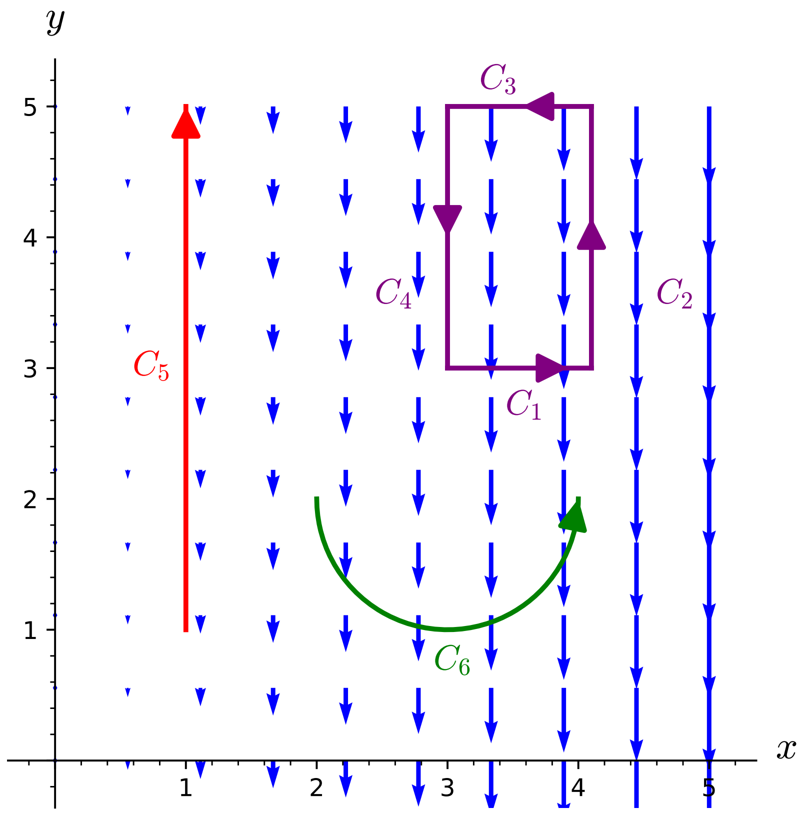


Figure 13.3.15 A vector field $\vec{F}$ and six oriented curves.

(a) Is $\int_{C_6} \vec{F} \cdot d\vec{r}$ positive, negative, or zero? Explain.

(b) Let $C = C_1 + C_2 + C_3 + C_4$. Determine if $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ is positive, negative, or zero.

(c) Order the line integrals below from smallest to largest.

$$\int_{C_1} \vec{F} \cdot d\vec{r} \quad \int_{C_2} \vec{F} \cdot d\vec{r} \quad \int_{C_3} \vec{F} \cdot d\vec{r} \quad \int_{C_4} \vec{F} \cdot d\vec{r} \quad \int_{C_5} \vec{F} \cdot d\vec{r}$$

Activity 13.3.5. Determine if the circulation of the vector field around each of the closed curves shown in Figure 13.3.16 is positive, negative, or zero.

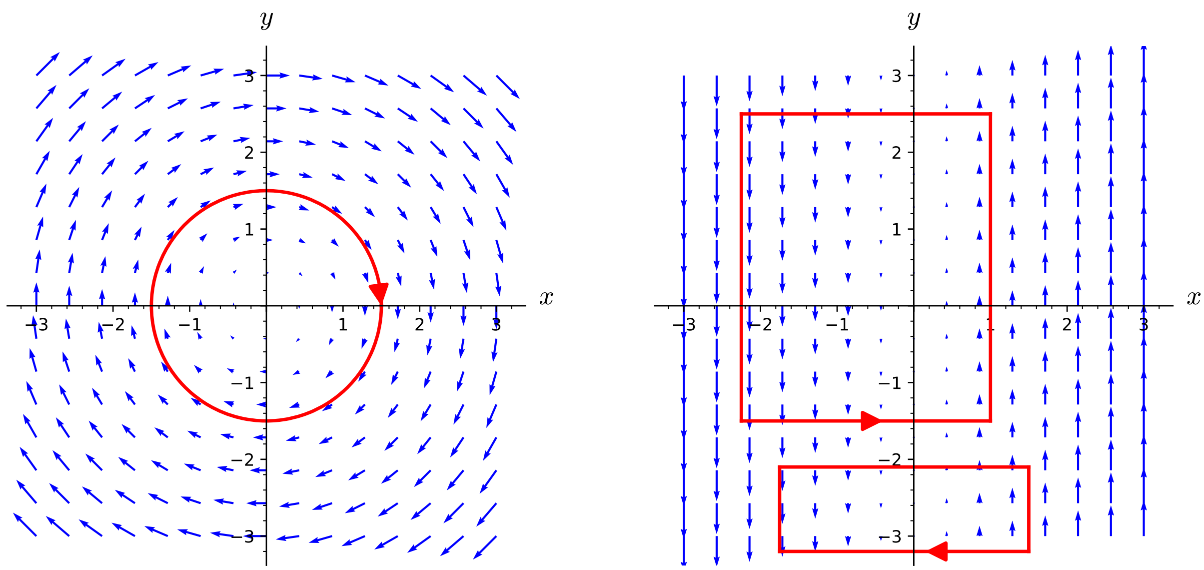


Figure 13.3.16 Vector fields and closed curves

13.4 Using Parameterizations to Calculate Line Integrals

Preview Activity 13.4.1. Let $\vec{F} = \langle xy, y^2 \rangle$, let C_1 be the line segment from $(1,1)$ to $(4,1)$, let C_2 be the line segment from $(4,1)$ to $(4,3)$, and let C_3 be the line segment from $(1,1)$ to $(4,3)$. Also let $C = C_1 + C_2$. This vector field and the curves are shown in Figure 13.4.1.

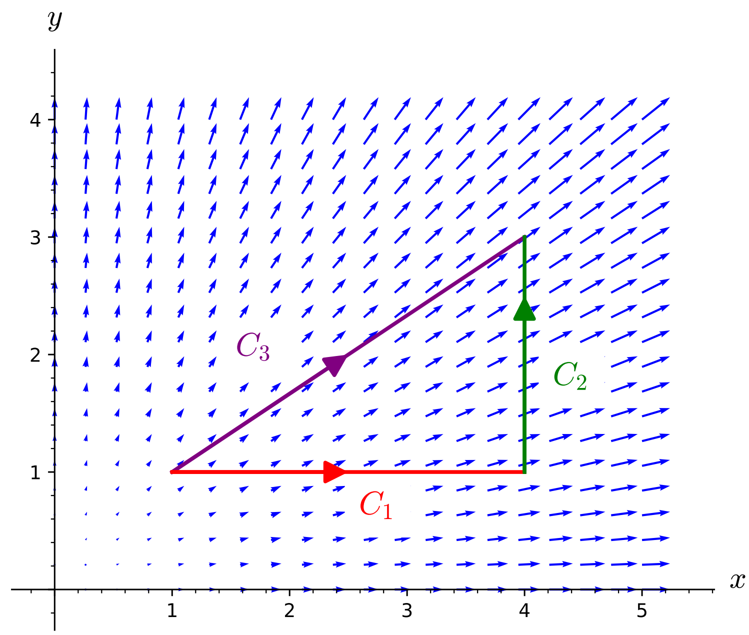


Figure 13.4.1 A vector field $\vec{F}$ and three oriented curves

(a) Every point along C_1 has $y = 1$. Therefore, along C_1, the vector field $\vec{F}$ can be viewed purely as a function of x. In particular, along C_1, we have $\vec{F}(x,1) = \langle x, 1 \rangle$. Since every point along C_2 has the same x-value, write $\vec{F}$ as a function of y only (for the points on C_2).

(b) Recall that $d\vec{r} \approx \Delta\vec{r}$, and along C_1, we have that $\Delta\vec{r} = \Delta x\hat{\imath} \approx dx\hat{\imath}$. Thus, $d\vec{r} = \langle dx, 0 \rangle$. We know that along C_1, $\vec{F} = \langle x, 1 \rangle$.

(i) Write $\vec{F} \cdot d\vec{r}$ along C_1 without using a dot product.

(ii) What interval of x-values describes C_1?

(iii) Write $\int_{C_1} \vec{F} \cdot d\vec{r}$ as an integral of the form $\int_a^b f(x)\,dx$ and evaluate the integral.

(c) Use an analogous approach to write $\int_{C_2} \vec{F} \cdot d\vec{r}$ as an integral of the form $\int_c^d g(y)\,dy$ and evaluate the integral.

(d) Use the previous parts and a property of line integrals to calculate $\int_C \vec{F} \cdot d\vec{r}$ without having to evaluate any additional integrals.

Activity 13.4.2. Is $\vec{F} = \langle xy, y^2 \rangle$ a gradient vector field? Why or why not?

Activity 13.4.3.

(a) Find the work done by the vector field $\vec{F}(x, y, z) = 6x^2 z\hat{\imath} + 3y^2\hat{\jmath} + x\hat{k}$ on a particle that moves from the point $(3, 0, 0)$ to the point $(3, 0, 6\pi)$ along the helix given by $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), t\rangle$.

(b) Let $\vec{F}(x, y) = \langle 0, x\rangle$. Let C be the closed curve consisting of the top half of the circle of radius 2 centered at the origin and the portion of the x-axis from $(2, 0)$ to $(-2, 0)$, oriented clockwise. Find the circulation of $\vec{F}$ around C.

Activity 13.4.4. Let $\vec{F}(x,y) = \langle y^2, 2xy+3\rangle$.

(a) Let C_1 be the portion of the graph of $y = 2x^3+3x^2-12x-15$ from $(-2,5)$ to $(3,30)$. Calculate $\int_{C_1} \vec{F}\cdot d\vec{r}$.

(b) Let C_2 be the line segment from $(-2,5)$ to $(3,30)$. Calculate $\int_{C_2} \vec{F}\cdot d\vec{r}$.

(c) Let C_3 be the circle of radius 3 centered at the origin, oriented counterclockwise. Calculate $\oint_{C_3} \vec{F}\cdot d\vec{r}$.

(d) To connect the previous parts of this activity, use a graphing utility to plot the curves C_1 and C_2 on the same axes.

(i) What type of curve is $C_1 - C_2$?

(ii) What is the value of $\oint_{C_1-C_2} \vec{F}\cdot d\vec{r}$?

(iii) What does your answer to part c allow you to say about the value of the line integral of $\vec{F}$ along the top half of C_3 compared to the line integral of $\vec{F}$ from $(3, 0)$ to $(-3, 0)$ along the bottom half of the circle of radius 3 centered at the origin?

Activity 13.4.5.

(a) The typical parametrization of the line segment from $(0,1)$ to $(3,3)$ (the oriented curve C_3 in Example 13.4.9) is $\vec{r}(t) = \langle 3t, 1+2t \rangle$ where $0 \leq t \leq 1$. Use this parametrization to calculate $\int_{C_3} \vec{F} \cdot d\vec{r}$ for the vector field $\vec{F} = x\hat{\imath}$ and compare your answer to the result of Example 13.4.9.

(b) Calculate $\int_C \langle (3xy + e^z), x^2, (4z + xe^z) \rangle \cdot d\vec{r}$ where C is the oriented curve consisting of the line segment from the origin to $(1,1,1)$ followed by the line segment from $(1,1,1)$ to $(0,0,2)$.

(c) Calculate $\int_{C'} \langle 3xy + e^z, x^2, 4z + xe^z \rangle \cdot d\vec{r}$ where C_3 is the line segment from $(0,0,0)$ to $(0,0,2)$.

(d) Is the vector field you considered in the previous two parts a gradient vector field? Why or why not? How does this compare to the vector field $\vec{F}$ of Activity 13.4.4?

13.5 Path-Independent Vector Fields and the Fundamental Theorem of Calculus for Line Integrals

Preview Activity 13.5.1. In Activity 13.4.4, we considered the vector field $\vec{F}(x, y) = \langle y^2, 2xy + 3\rangle$ and two different oriented curves from $(-2, 5)$ to $(3, 30)$. We found that the value of the line integral of $\vec{F}$ was the same along those two oriented curves.

(a) Verify that $\vec{F}(x, y) = \langle y^2, 2xy + 3\rangle$ is a gradient vector field by showing that $\vec{F} = \nabla f$ for the function $f(x, y) = xy^2 + 3y$.

(b) Calculate the change in the output of the scalar function f over the curves C_1 and C_2. In other words, what is the difference in the output of f at the start of the curve and the end of the curve? How does this value compare to the value of the line integral $\int_{C_1} \vec{F} \cdot d\vec{r}$ you found in Activity 13.4.4?

(c) Let C_3 be the line segment from $(1, 1)$ to $(3, 4)$. Calculate $\int_{C_3} \vec{F} \cdot d\vec{r}$ as well as $f(3, 4) - f(1, 1)$. Write a sentence that compares your answer to this part to your result for part 13.5.1.b.

Activity 13.5.2. Calculate each of the following line integrals.

(a) $\int_C \nabla f \cdot d\vec{r}$ if $f(x,y) = 3xy^2 - \sin(x) + e^y$ and C is the top half of the unit circle oriented from $(-1,0)$ to $(1,0)$.

(b) $\int_C \nabla g \cdot d\vec{r}$ if $g(x,y,z) = xz^2 - 5y^3\cos(z) + 6$ and C is the portion of the helix $\vec{r}(t) = \langle 5\cos(t), 5\sin(t), 3t\rangle$ from $(5,0,0)$ to $(0,5,9\pi/2)$.

(c) $\int_C \nabla h \cdot d\vec{r}$ if $h(x,y,z) = 3y^2e^{y^3} - 5x\sin(x^3z) + z^2$ and C is the curve consisting of the line segment from $(0,0,0)$ to $(1,1,1)$, followed by the line segment from $(1,1,1)$ to $(-1,3,-2)$, followed by the line segment from $(-1,3,-2)$ to $(0,0,10)$.

Activity 13.5.3. Let $\vec{G}(x, y, z) = \langle 3e^{y^2} + z\sin(x), 6xye^{y^2} - z, 3z^2 - y - \cos(x)\rangle$ and $\vec{H}(x, y, z) = \langle 3x^2y, x^3 + 2yz^3, xz + 3y^2z^2\rangle$.

(a) If $\vec{G}$ and $\vec{H}$ are to be gradient vector fields, then there are functions g and h for which $\vec{G} = \nabla g$ and $\vec{H} = \nabla h$. If such functions g and h exist, what would g_y, g_z, h_x, h_y, and h_z be?

(b) Let $g_1(x, y, z) = 3xe^{y^2} + xyz - z\sin(x)$. Calculate $\partial g_1/\partial x$. Could g_1 be a potential function for the vector field $\vec{G}$?

(c) Find a function g so that $\partial g/\partial x = 3e^{y^2} + z\sin(x)$. Find a function h so that $\partial h/\partial x = 3x^2y$.

(d) When finding the most general anti-derivative for a function of one variable, we add a constant of integration (usually denoted by C) to capture the fact that any constant will become 0 through differentiation.

(i) When taking the partial derivative with respect to x of a function of x, y, and z, what variables can appear in terms that become 0 in the partial derivative because they are treated as constants?

(ii) What does this tell you should be added to g and h in the previous part to make them the most general possible functions with the desired partial derivatives with respect to x?

(e) Now calculate $\partial g/\partial y$ and $\partial h/\partial y$ based on your choices for part 13.5.3.c. Write a few sentences to explain why this tells you that we must have

$$g(x, y, z) = 3xe^{y^2} - z\cos(x) - yz + m_1(z)$$

and

$$h(x, y, z) = x^3 y + y^2 z^3 + m_2(z)$$

for some functions m_1 and m_2 depending only on z.

(f) Calculate $\frac{\partial g}{\partial z}$ and $\frac{\partial h}{\partial z}$ for the functions in the part above. Notice that m_1 and m_2 are functions of z alone, so taking a partial derivative with respect to z is the same as taking an ordinary derivative, and thus you may use the notation $m_1'(z)$ and $m_2'(z)$.

(g) Explain why $\vec{G}$ is a gradient vector field but $\vec{H}$ is not a gradient vector field. Find a potential function for $\vec{G}$.

Activity 13.5.4. Calculate each of the following line integrals.

(a) $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F}(x, y) = \langle 2x, 2y \rangle$ and C is the line segment from $(1, 2)$ to $(-1, 0)$.

(b) $\int_C \vec{G} \cdot d\vec{r}$ if $\vec{G}(x, y) = \langle 4x^3 - 12y\cos(xy), 9y^2 - 12x\cos(xy) \rangle$ and C is the portion of the unit circle from $(0, -1)$ to $(0, 1)$.

(c) $\int_C \vec{H} \cdot d\vec{r}$ if $\vec{H}(x, y, z) = \langle H_1, H_2, H_3 \rangle$ with

$$\begin{aligned} H_1(x, y, z) &= e^{z^2} + 2xy^3z + \cos(x) - y^3\sin(x) \\ H_2(x, y, z) &= 2ye^{y^2} + 3x^2y^2z + 3y^2z^2 + 3y^2\cos(x) \\ H_3(x, y, z) &= x^2y^3 + 2xze^{z^2} + 2y^3z - 4z^3 \end{aligned}$$

and C is the curve consisting of the line segment from $(1, 1, 1)$ to $(3, 0, 3)$, followed by the line segment from $(3, 0, 3)$ to $(1, 5, -1)$, followed by the line segment from $(1, 5, -1)$ to $(0, 0, 0)$.

Activity 13.5.5. Suppose that $\vec{F}$ is a continuous path-independent vector field (in $\mathbb{R}^2$ or $\mathbb{R}^3$) on some region D.

(a) Let P and Q be points in D and let C_1 and C_2 be oriented curves from P to Q. What can you say about $\int_{C_1} \vec{F} \cdot d\vec{r}$ and $\int_{C_2} \vec{F} \cdot d\vec{r}$?

(b) Let $C = C_1 - C_2$. Explain why C is a closed curve.

(c) Calculate $\oint_C \vec{F} \cdot d\vec{r}$.

(d) Write a sentence that summarizes what we can conclude about line integrals of $\vec{F}$ at this point in the activity.

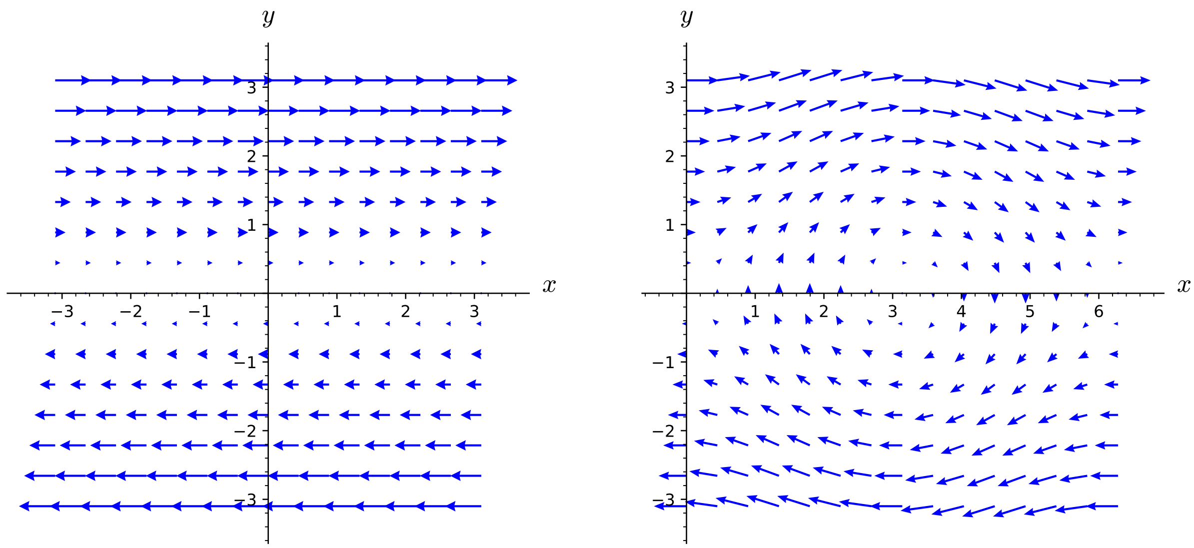
(e) Now let us suppose that $\vec{G}$ is a continuous vector field on a region D for which $\oint_C \vec{G} \cdot d\vec{r} = 0$ for all closed curves C. Pick two points P and Q in D. Let C_1 and C_2 be oriented curves from P to Q. What type of curve is $C = C_1 - C_2$?

(f) What is $\oint_C \vec{G} \cdot d\vec{r}$? Why?

(g) What does that tell you about the relationship between $\int_{C_1} \vec{G} \cdot d\vec{r}$ and $\int_{C_2} \vec{G} \cdot d\vec{r}$?

(h) Explain why this shows that $\vec{G}$ is path-independent.

Activity 13.5.6. Explain why neither of the vector fields in Figure 13.5.4 is path-independent.



(a) $\vec{F}$ **(b)** $\vec{G}$

Figure 13.5.4 Two vector fields that are not path-independent.

Activity 13.5.7. Let $\vec{F} = \langle F_1, F_2 \rangle$ be a continuous, path-independent vector field on an open, path-connected region D. We will assume that D is in $\mathbb{R}^2$ and $\vec{F}$ is a two-dimensional vector field, but the ideas below generalize completely to $\mathbb{R}^3$. We want to define a function f on D by using the vector field $\vec{F}$ and line integrals, much like the Second Fundamental Theorem of Calculus allows us to define an antiderivative of a continuous function using a definite integral. To that end, we assign $f(x_0, y_0)$ an arbitrary value. (Setting $f(x_0, y_0) = 0$ is probably convenient, but we won't explicitly tie our hands. Just assume that $f(x_0, y_0)$ is defined to be some number.) Now for any other point (x, y) in D, define

$$f(x, y) = f(x_0, y_0) + \int_C \vec{F} \cdot d\vec{r},$$

where C is any oriented path from (x_0, y_0) to (x, y). Since D is path-connected, such an oriented path must exist. Since $\vec{F}$ is path-independent, f is well-defined. If different paths from (x_0, y_0) to (x, y) gave different values for the line integral, then we would not be sure what $f(x, y)$ really is.
To better understand this mysterious function f we've now defined, let's start looking at its partial derivatives.

(a) Since D is open, there is a disc (perhaps very small) surrounding (x, y) that is contained in D, so fix a point (a, b) in that disc. Since D is path-connected, there is a path C_1 from (x_0, y_0) to (a, b). Let C_y be the line segment from (a, b) to (a, y) and let C_x be the line segment from (a, y) to (x, y). (See Figure 13.5.6.) Rewrite $f(x, y)$ as a sum of $f(x_0, y_0)$ and line integrals along C_1, C_y, and C_x.

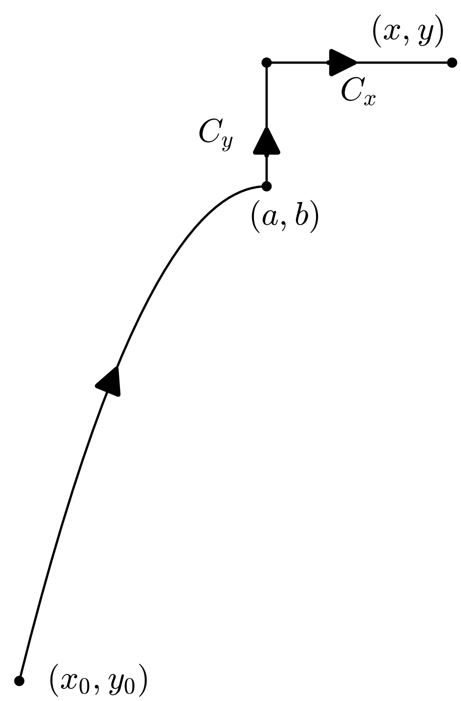


Figure 13.5.6 A piecewise smooth oriented curve from (x_0, y_0) to (x, y).

(b) Notice that we can parametrize C_y by $\langle a, t\rangle$ for $b \le t \le y$. Find a similar parametrization for C_x.

(c) Use the parametrization from above to write $\int_{C_y} \vec{F} \cdot d\vec{r}$ and $\int_{C_x} \vec{F} \cdot d\vec{r}$ as single variable integrals in the manner of Section 13.4. Use the fact that $\vec{F}(x, y) = \langle F_1(x, y), F_2(x, y)\rangle$ to express your integrals in terms of F_1 and F_2 without any dot products.

(d) Rewrite your expression for $f(x, y)$ using a line integral along C_1 and the single variable integrals above.

(e) Notice that your expression for $f(x, y)$ from the previous part only depends on x as the upper limit of an single variable integral. Use the Second Fundamental Theorem of Calculus to calculate $f_x(x, y)$.

(f) To calculate $f_y(x, y)$, we continue to consider a path C_1 from (x_0, y_0) to (a, b), but now let L_x be the line segment from (a, b) to (x, b) and let L_y be the line segment from (x, b) to (y, b). Modify the process you used to find $f_x(x, y)$ to find $f_y(x, y)$.

(g) What can you conclude about the relationship between ∇f and $\vec{F}$? What does this tell you about $\vec{F}$ beyond that it is path-independent and continuous?

13.6 Line Integrals of Scalar Functions

Preview Activity 13.6.1. In order to pay for tuition, you take a job driving a mining machine that collects a very valuable mineral called copium. Copium is only produced on the surface and is mined by scooping up the soil at the front of your machine, so the amount of copium ore collected depends on the density of the ore and the distance driven by the mining machine. The plot of land you are mining has been surveyed for the density of copium ore and is presented in the contour plot below.

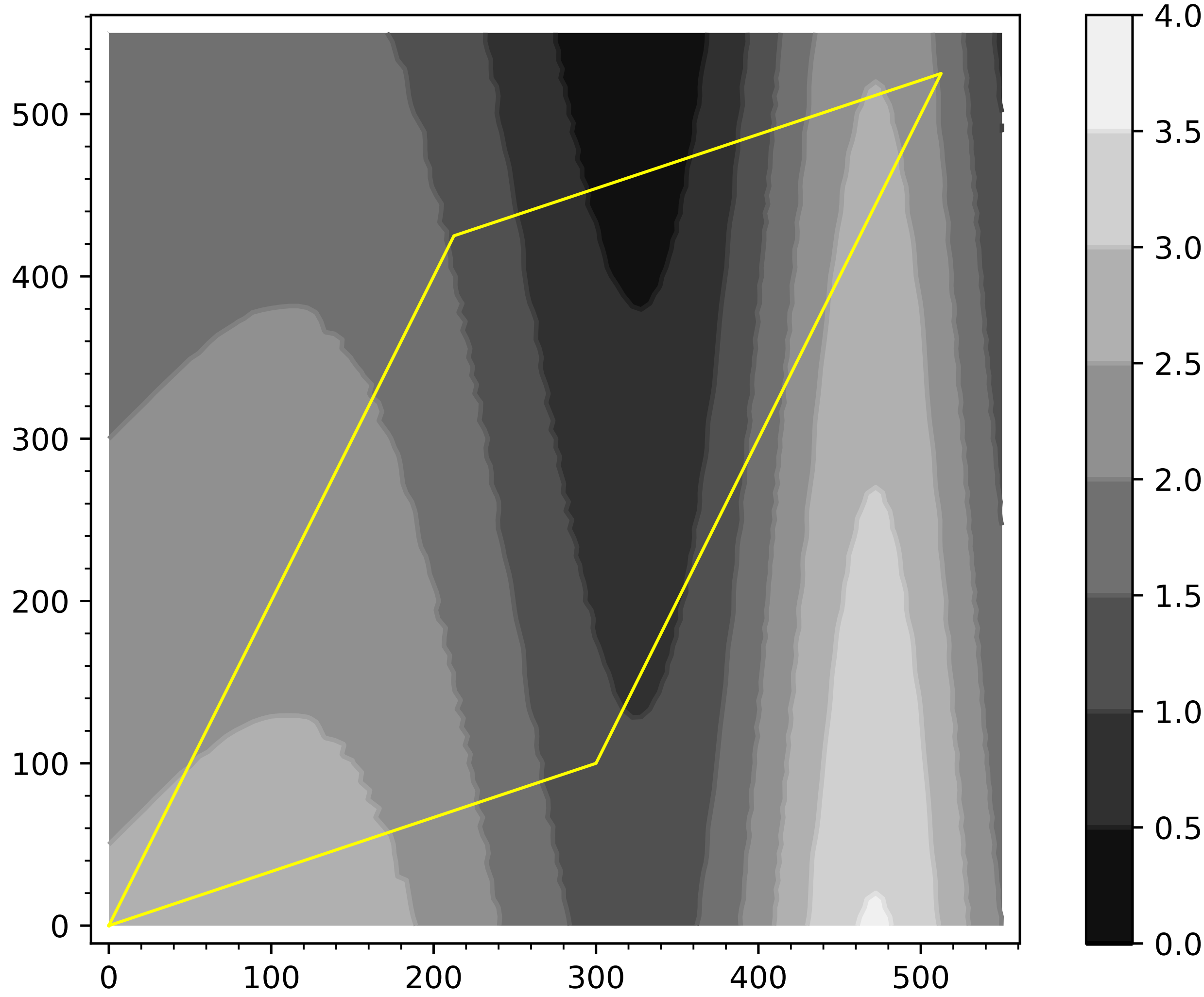


Figure 13.6.1 A plot of land with density of copium deposits

(a) Estimate the amount of copium that would be mined from driving along the left side of plot. You should write a few sentences about how you got your estimate based on the copium density and length of the path. (Did you use more than one piece?)

(b) Estimate the amount of copium that would be mined by driving along the entire outer edge of the plot. You should write a few sentences about how you got your estimate based on the copium density and length of the paths.

(c) Estimate the amount of copium that would be mined from the scraping the curved path shown in Figure 13.6.2. You should use at least 3 segments in your estimate. You should write a few sentences about how you got your estimate based on the copium density and length of the paths.

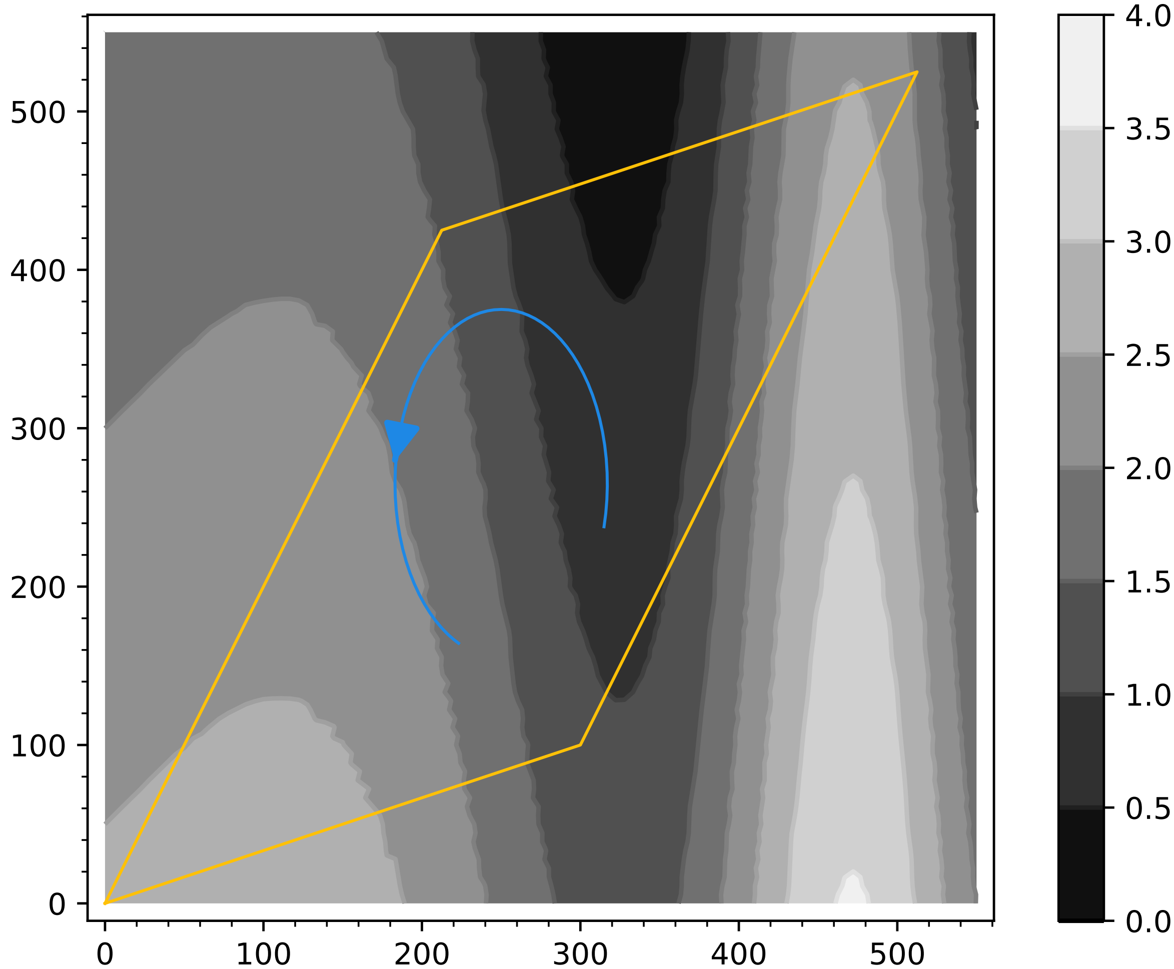


Figure 13.6.2 A plot of land with copium Density and a path plotted in blue

(d) Estimate the amount of copium that would be mined from the scraping the curved path shown below. You should use at least 3 segments in your estimate. You should also explain how and why your answer to this question is different or similar to the previous task.

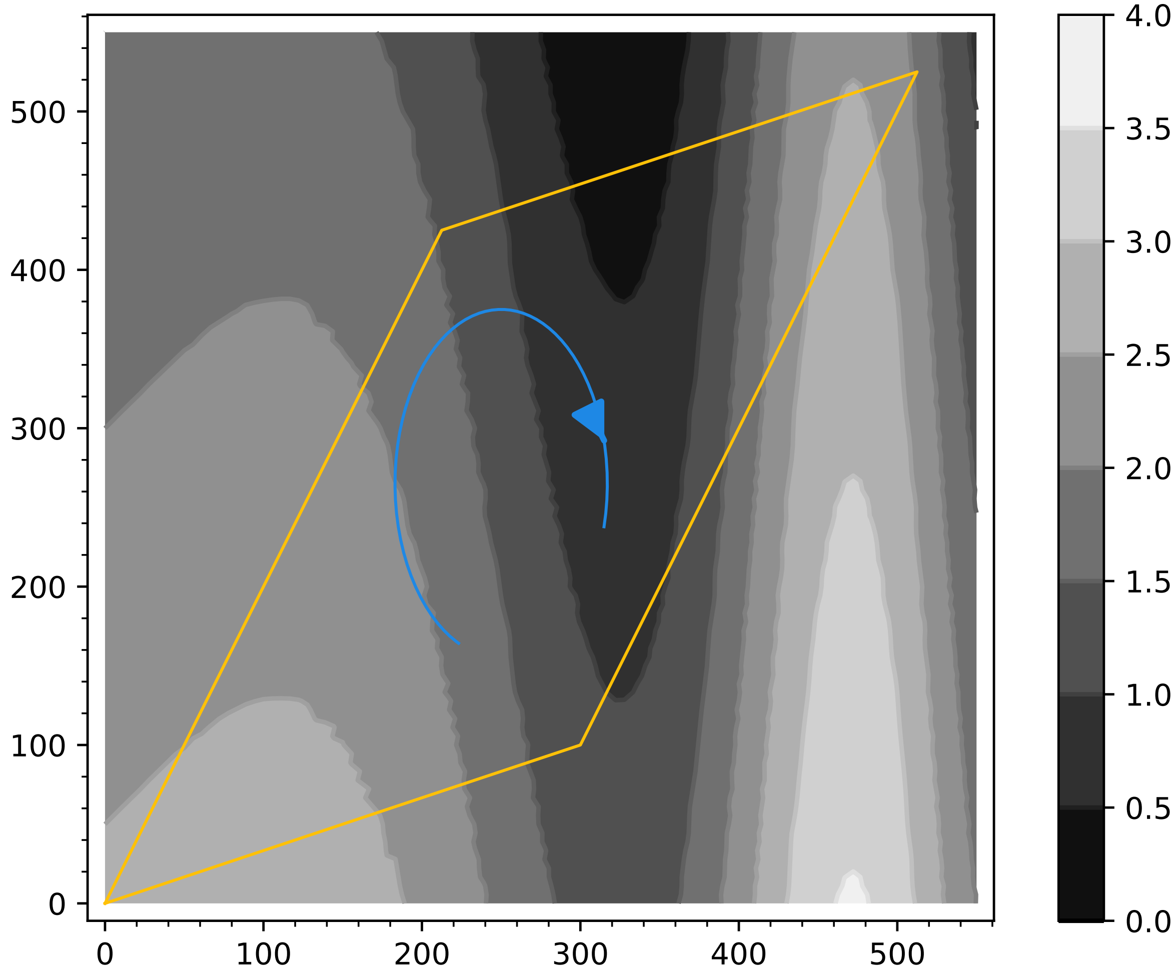


Figure 13.6.3 A plot of land with copium density and a path plotted in blue

Activity 13.6.2. In this activity, we will be making sense of scalar line integrals by examining a few common functions and justifying whether the scalar line integrals given are positive, negative, or zero. Let the functions f_1, f_2, f_3, and f_4 be defined as

- $f_1(x, y, z) = y$
- $f_2(x, y, z) = z$
- $f_3(x, y, z) = x^2$
- $f_4(x, y, z) = x - y$

(a) For each of the paths given below, sketch (in either 2D or 3D) the curve and label at least three points on the curve including the end points (if they exist).

(i) C_1 is the part of the unit circle in the xy-plane centered at the origin that is above the line $y = -x$.

(ii) C_2 is the part of the curve at the intersection of the cylinder given by $x^2 + y^2 = 1$ and the plane $z = x$ such that $y \geq -x$; You may want to consider the circle that is the intersection of $x^2 + y^2 = 1$ and $z = x$, then think about which half of this circle satisfies the inequality $y \geq -x$

(iii) C_3 is the part of the helix given by $\vec{r}(t) = \langle \cos(t), \sin(t), \frac{t}{2\pi} \rangle$ with $t \in [0, \pi]$

(b) For each of the functions f_1, f_2, and f_3 defined above, state whether $\int_{C_1} f_i \, ds$ is positive, negative, or zero. Be sure to justify your answer in terms of the function being integrated *and* the particulars of the curve of integration.

(c) For each of the functions f_1, f_2, and f_3, defined above, state whether $\int_{C_2} f_i \, ds$ is positive, negative, or zero. Be sure to justify your answer in terms of the function being integrated *and* the particulars of the curve of integration.

(d) For each of the functions f_1, f_2, and f_3, defined above, state whether $\int_{C_3} f_i \, ds$ is positive, negative, or zero. Be sure to justify your answer in terms of the function being integrated *and* the particulars of the curve of integration.

(e) For the function f_4, defined above, state each of the following integrals is positive, negative, or zero. Be sure to justify your answer in terms of the function being integrated *and* the particulars of the curve of integration. You should consider which parts of the curve being integrated will have positive/negative/zero output for the function f_4.

$$\int_{C_1} f_4 \, ds$$
$$\int_{C_2} f_4 \, ds$$
$$\int_{C_3} f_4 \, ds,$$

Activity 13.6.3. In this activity, we will examine why we must be careful when using symmetry to make arguments about scalar line integral. Let C_1 and C_2 be the paths shown in Figure 13.6.16. We will consider the function $f(x, y) = x$ for this activity.

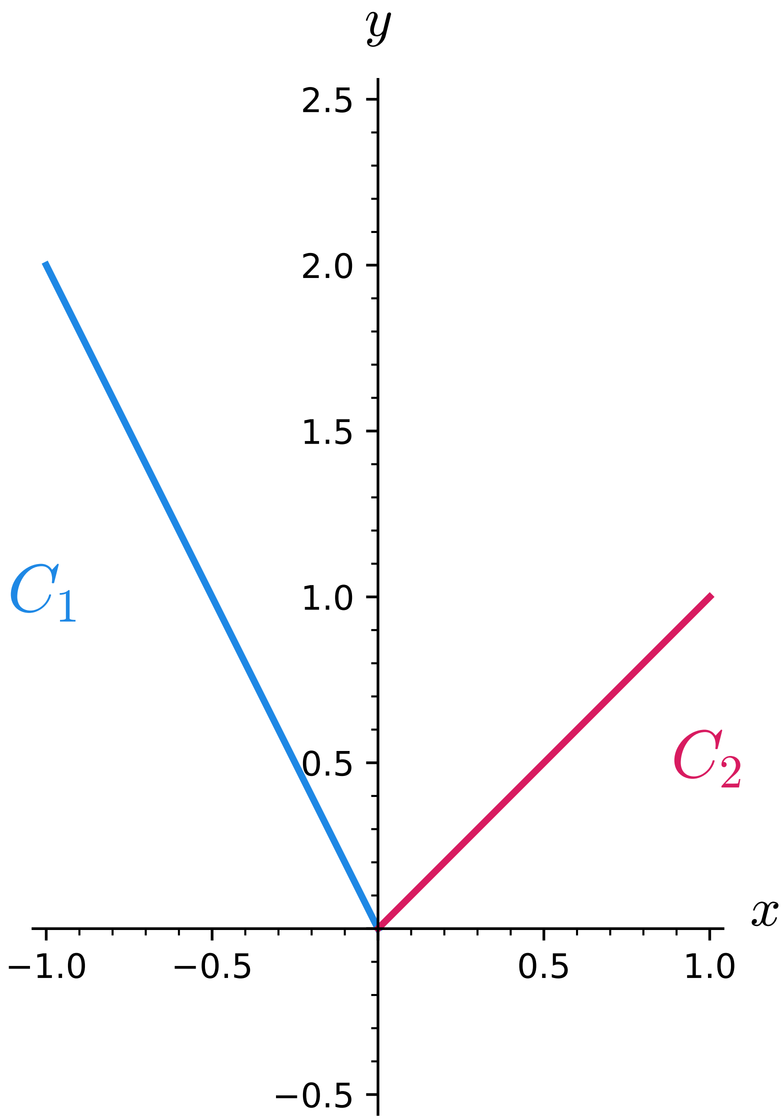


Figure 13.6.16 A plot of paths C_1 and C_2

(a) Parameterize C_1 and C_2 as $\vec{r}_1(t)$ and $\vec{r}_2(t)$. (It is fine to have $0 \leq t \leq 1$ for both of your parameterizations.)

(b) Use Theorem 13.6.11 to compute $\int_{C_1} f\,ds$ and $\int_{C_2} f\,ds$.

(c) As with line integrals of vector fields, we have that $\int_{C_1+C_2} f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$. Use this property to compute $\int_{C_1+C_2} f\,ds$.

Activity 13.6.4 Explaining Properties of Scalar Line Integrals.. In this activity, we will be explaining each of the Properties from Properties of Scalar Line Integrals in the context of our copium mining analogy from Preview Activity 13.6.1. Remember that the curve in our scalar line integral corresponds to the path the mining rig will take and the function in the scalar line integral measures the density of copium at that point on the surface.

(a) Explain in your own words what $\displaystyle\int_C f\,ds$ means in the copium analogy and what exactly would be measured by this scalar line integral.

(b) Explain in your own words what $\displaystyle\int_C (kf)\,ds = k\int_C f\,ds$ means in the copium analogy. It may be helpful to describe each side of the equation separately and say why they are equal in the analogy.

(c) Explain in your own words what $\displaystyle\int_C (f+g)\,ds = \int_C f\,ds + \int_C g\,ds$ means in the copium analogy. It may be helpful to describe each side of the equation separately and say why they are equal in the analogy. You may also find it helpful to consider f and g to measure density of different materials.

(d) Explain in your own words what $\displaystyle\int_{-C} f\,ds = \int_C f\,ds$ means in the copium analogy. It may be helpful to describe each side of the equation separately and say why they are equal in the analogy.

(e) Explain in your own words what $\int_{C_1+C_2} f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$ means in the copium analogy. It may be helpful to describe each side of the equation separately and say why they are equal in the analogy.

13.7 The Divergence of a Vector Field

Preview Activity 13.7.1. In this preview activity, we will look at several two-dimensional vector fields and try to assess when the vector field has increased or decreased in strength over a given region. We begin with graphs of the three vector fields, $\vec{F}$, $\vec{G}$, and $\vec{H}$. Parts a, b, and c ask you to answer the same three questions about the vector field and square illustrated in each of the figures. Part d asks you to think further about the third vector field.

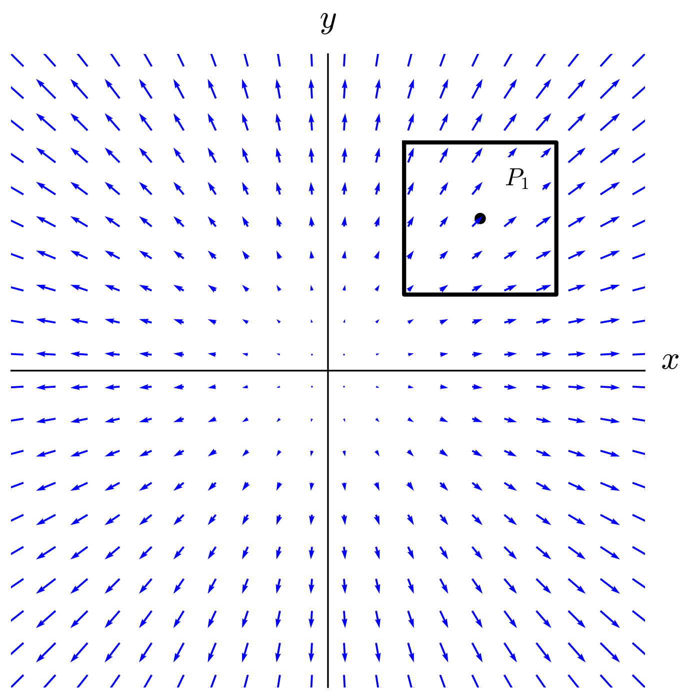


Figure 13.7.2 Vector Field $\vec{F}$

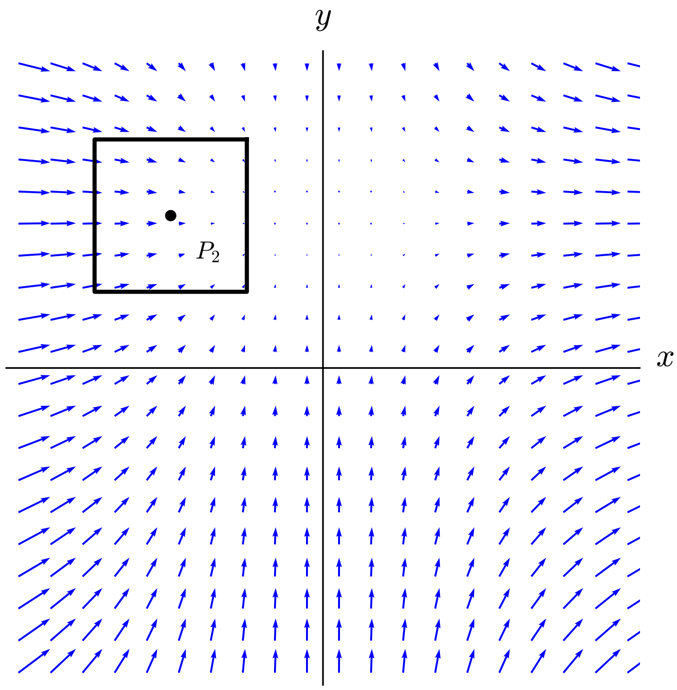


Figure 13.7.3 Vector Field $\vec{G}$

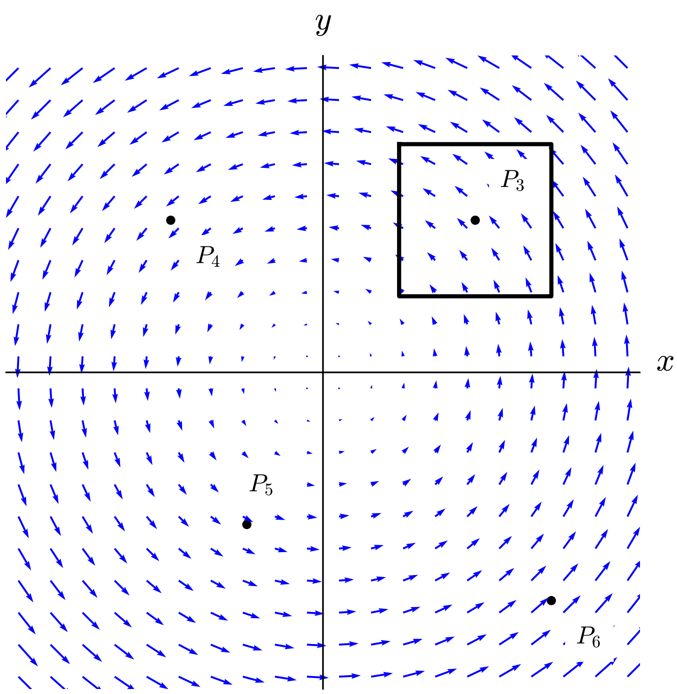


Figure 13.7.4 Vector Field $\vec{H}$

(a) For each of the vector fields $\vec{F}$, $\vec{G}$, and $\vec{H}$ and the square centered on P_1, P_2, and P_3 (respectively), which statement do you think best applies?

i More of the vector field is going into the square than going out.

ii Less of the vector field is going into the square than going out.

iii The same amount of the vector field is going into the square as is going out.

(b) For each of the vector fields $\vec{F}$, $\vec{G}$, and $\vec{H}$ (and corresponding square), does your answer to part a suggest that the vector field is being created, destroyed, or is unchanging in strength inside the square? Write a sentence to explain your thinking for each vector field.

(c) Would the answer to parts a or b change if you used a smaller square centered on P_1, P_2, and P_3 for the corresponding vector fields? Write a sentence to explain your thinking for each vector field.

(d) Thinking now only about the vector field $\vec{H}$, would your answers to parts a, b, or c change if you considered squares around points P_4, P_5, or P_6? Write a couple of sentences to explain your thinking.

Activity 13.7.2 Graphical Representations of Divergence..

(a) For this part of the activity, consider the vector field $\vec{F}$ shown in Figure 13.7.12.

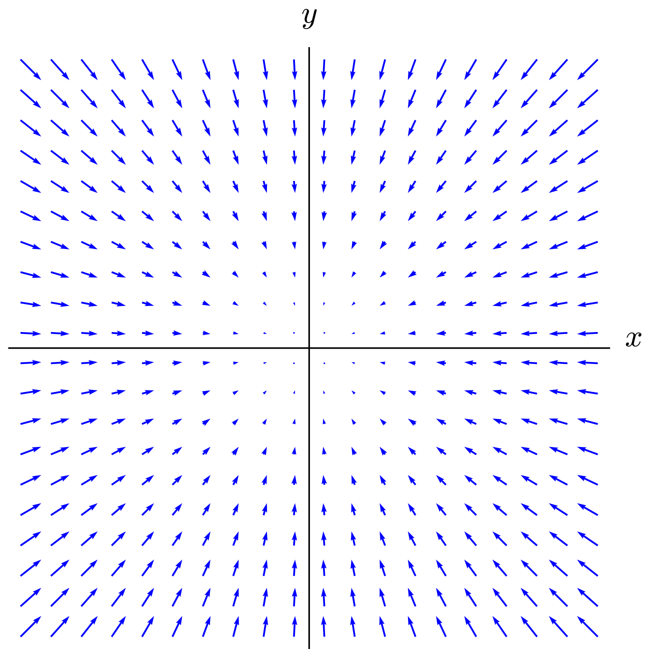


Figure 13.7.12 Vector field $\vec{F}$

(i) Draw a circle in the first quadrant of the vector field $\vec{F}$ depicted in Figure 13.7.12. Based on the flow of the vector field into or out of the circle, do you think the vector field is increasing in strength, decreasing in strength, or not changing in overall strength in the first quadrant?

(ii) As you move along the flow of the vector field in the first quadrant of Figure 13.7.12, does your vector field increase in magnitude, decrease in magnitude, or have constant magnitude?

(iii) Draw a circle in each of quadrants II, III, and IV. Based on the flow of the vector field into or out of your circles, do you think the vector field is increasing in strength, decreasing in strength, or not changing in overall strength in quadrants III, II, and IV?

(iv) As you move along the flow of the vector field in the third quadrant of Figure 13.7.12, does your vector field increase in magnitude, decrease in magnitude, or have constant magnitude.

(v) Based on your arguments above, describe why the divergence of $\vec{F}$ is negative for all points in the xy-plane.

(b) Look at the plot of the vector field $\vec{G}$ in Figure 13.7.13 and state whether you think the vector field is increasing in strength, decreasing in strength, or not changing in overall strength in each of the four quadrants. You can make your argument in terms of the change in magnitude along the flow of the vector field or in terms of the net flow into or out of a small region on the plane. You may need to make separate arguments for each of the four quadrants.

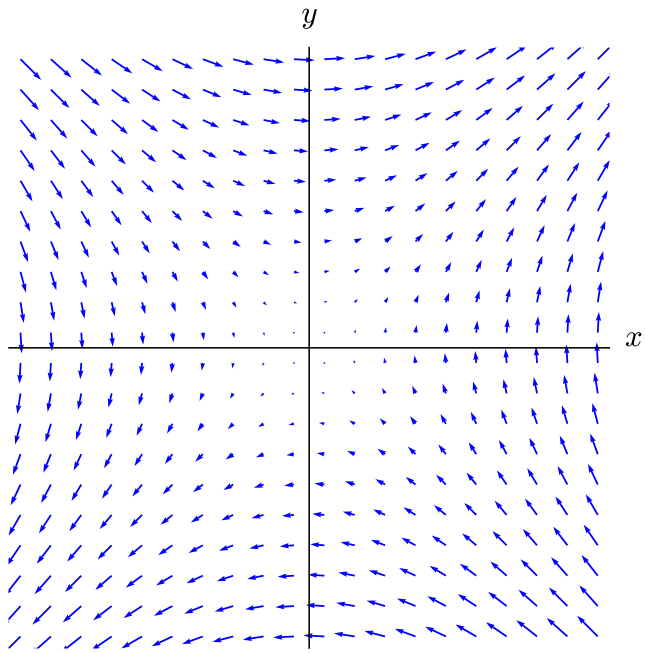


Figure 13.7.13 Vector field $\vec{G}$

(c) Look at the plot of the vector field $\vec{H}$ in Figure 13.7.14 below and state whether you think the vector field is increasing in strength, decreasing in strength, or not changing in overall strength in each of the four quadrants. You can make your argument in terms of the change in magnitude along the flow of the vector field or in terms of the net flow into or out of a small region on the plane. You may need to make separate arguments for each of the four quadrants.

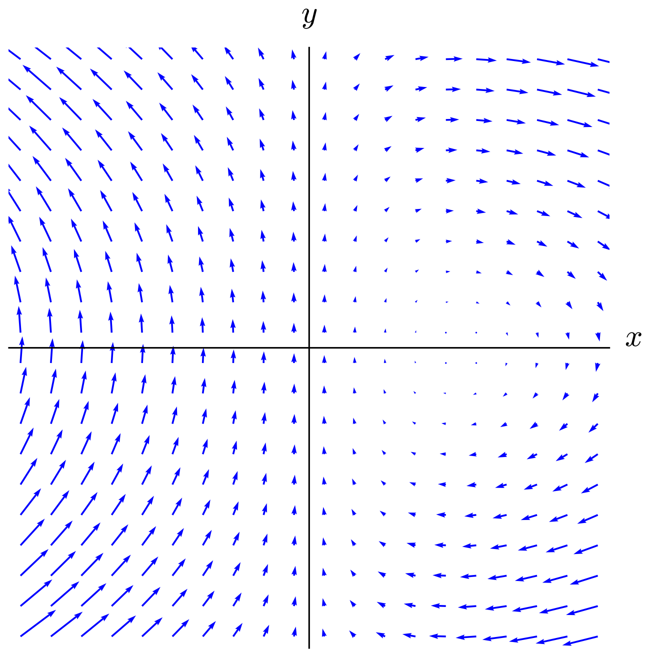


Figure 13.7.14 Vector field $\vec{H}$

Activity 13.7.3.

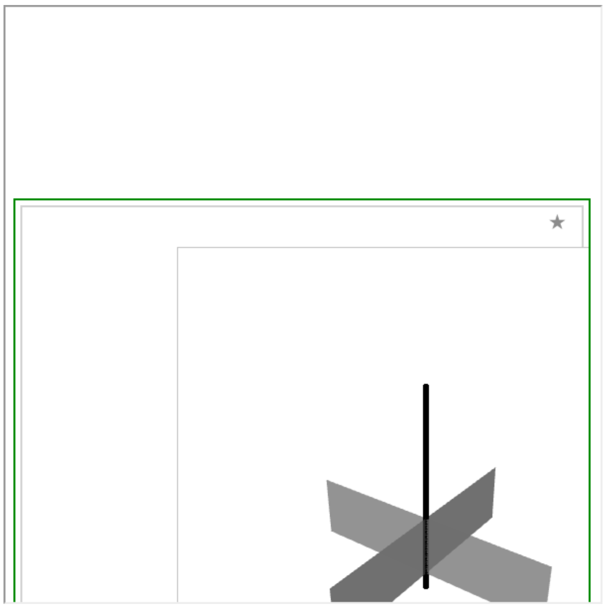
(a) Calculate the divergence of the vector fields given below.

- $\vec{F}(x, y) = \langle -x, -y \rangle$
- $\vec{G}(x, y) = \langle y, x \rangle$
- $\vec{H}(x, y) = \langle xy, 1 - x \rangle$

(b) Explain how your answers to the questions in Activity 13.7.2 can be explained by using your results from part a of this activity.

13.8 The Curl of a Vector Field

Preview Activity 13.8.1. We would like to understand and measure rotation of a vector field near a particular point. In order to investigate this concept, we will look at some two-dimensional vector fields and think about whether the vector field shown will rotate a small pinwheel or spinner placed at a particular location. The sort of spinner we imagine is illustrated in Figure 13.8.3. It consists of a central axis with a four-bladed paddle placed at one end of the axis. We imagine that the spinner is anchored at a point and the vector field, perhaps thought of as a fluid flow or wind velocity vector field, pushes against the blades of the spinner's paddle. In this activity, we will be trying to assess how the the spinner will rotate around the black axel (as an axis of rotation).



Standalone

In Context

Embed

Figure 13.8.3 A paddle-bladed spinner

To begin our investigation of the rotation of a spinner in a vector field, we will look at the vector field $\vec{F}$ in Figure 13.8.4. As you think about these questions, draw an "X" at each of the points about which you are asked and consider the vector field as being the pattern of a wind blowing across the plane.

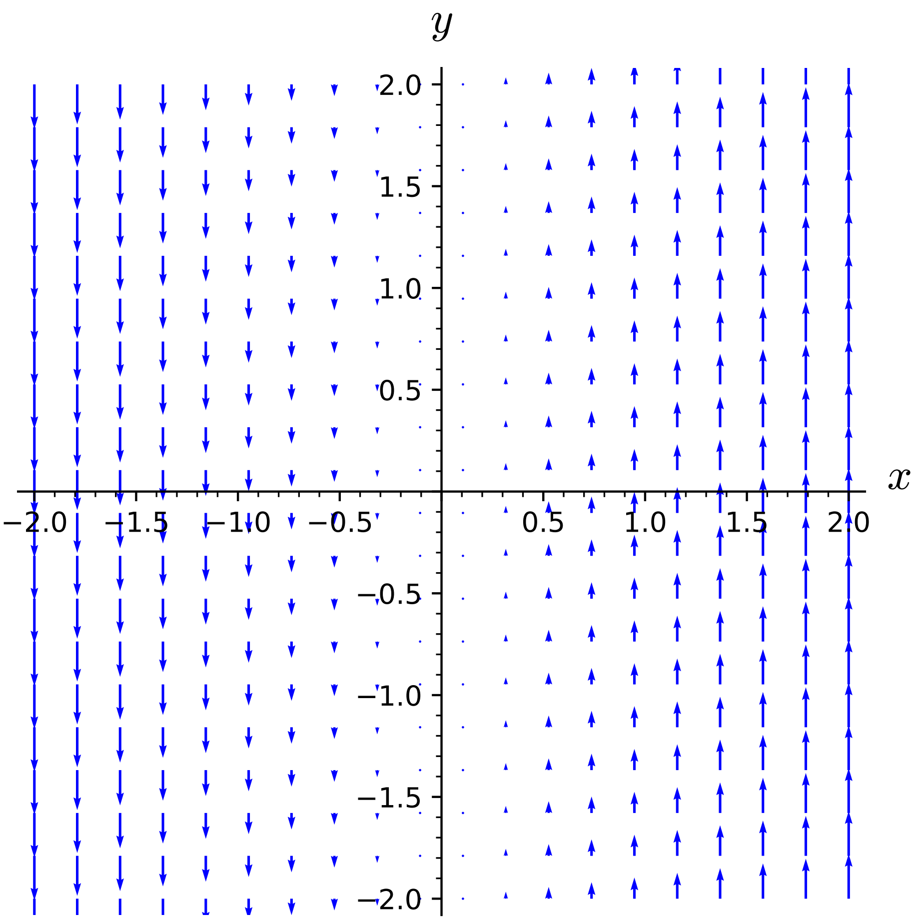


Figure 13.8.4 The vector field $\vec{F}$

(a) Draw an X at the origin to act as your spinner. Draw a vector on the top right blade of your spinner that represents how the wind will push on that blade. Next, draw a vector on each of the other blades of your spinner that represents how the wind will push on that blade.

(b) Use your vector representations from the previous part to describe if a small spinner placed at the origin would spin clockwise, counterclockwise, or not spin at all.

(c) Now we would like to look at the rotational strength of $\vec{F}$ at the point $(1, 1)$. As before, draw a spinner at this point and draw vectors on each of the blades to represent how the wind will push on that blade. It is important at this stage to draw the relative lengths of the vectors on each blade to scale so you can see which blades have a larger force due to the wind.

(d) Use your vector representations from the previous part to describe if a small spinner placed at at $(1, 1)$ would spin clockwise, counterclockwise, or not spin at all. You should pay attention to which blades will have a stronger force (in comparison to the blade on the other side).

(e) Would a small spinner placed at $(-2, 1)$ spin clockwise, counterclockwise, or not spin at all? Draw a representation of a spinner if necessary to demonstrate your ideas.

(f) Are there any points on the xy-plane where the spinner would not turn counterclockwise?

Activity 13.8.2 Matching Visual Measurements to Algebraic Calculations.. Remember that the circulation density of a vector field $\vec{F} = \langle F_1, F_2 \rangle$ at a point (a, b) is calculated as

$$\text{Circulation Density at the point } (a,b) = \frac{\partial F_2}{\partial x}(a,b) - \frac{\partial F_1}{\partial y}(a,b).$$

The circulation density will be positive when a small spinner placed at (a, b) will spin counterclockwise, negative when the spinner will move clockwise, and zero when the spinner does not rotate.
For each of the vector fields given below, answer the following questions:

i. What is the formula for the circulation density of the given vector field?

ii. For what points will you have a positive circulation density?

iii. For what points will you have a negative circulation density?

iv. For what points will you have a zero circulation density?

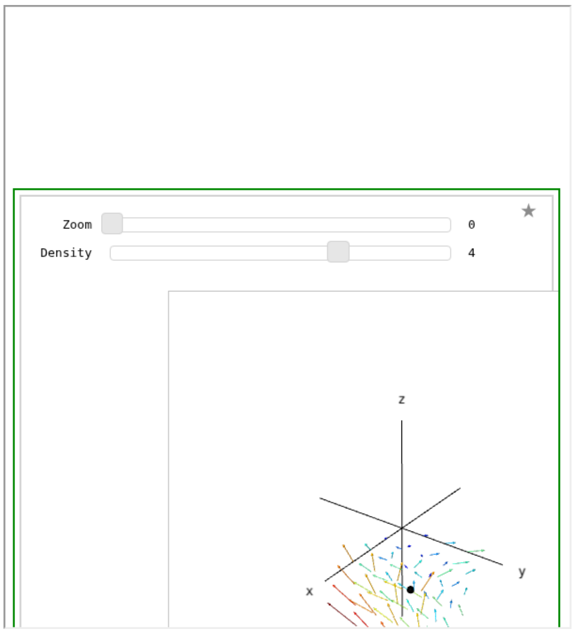
(a) $\langle -y, x \rangle$

(b) $\langle x, y \rangle$

(c) $\langle 1 - x, xy \rangle$

Activity 13.8.3 Estimating Curl in Three Dimensions..

(a) Consider the vector field $\vec{F}$ plotted in Figure 13.8.18. You can adjust the size of the region around $(1,1,-2)$ over which the vector field is plotted using the "Zoom" slider. The "Density" slider allows you to adjust the number of vectors plotted. Try to identify any rotation in the three dimensional vector field plot at the point $(1,1,-2)$. Write a sentence describing how a spinner placed at $(1,1,-2)$ would rotate, including along which axis it would rotate. Try to state your answer as a vector representing the rotational strength of the vector field at $(1,1,-2)$.



Standalone

In Context

Embed

Figure 13.8.18 A vector field plotted in a region around the point $(1,1,-2)$

(b) You likely found it difficult to decide how you thought a spinner might rotate in this new, three-dimensional setting. Let's look at the vector field in the plane $z=-2$, as displayed in Figure 13.8.19. Do you think a spinner placed on the red point would rotate clockwise, counterclockwise, or not rotate? If the spinner will rotate, you should think about what the axis of rotation would be and whether the rotation should be positive or negative. Summarize your result as a vector representing the rotational strength of $\vec{F}$ in the plane $z=-2$.

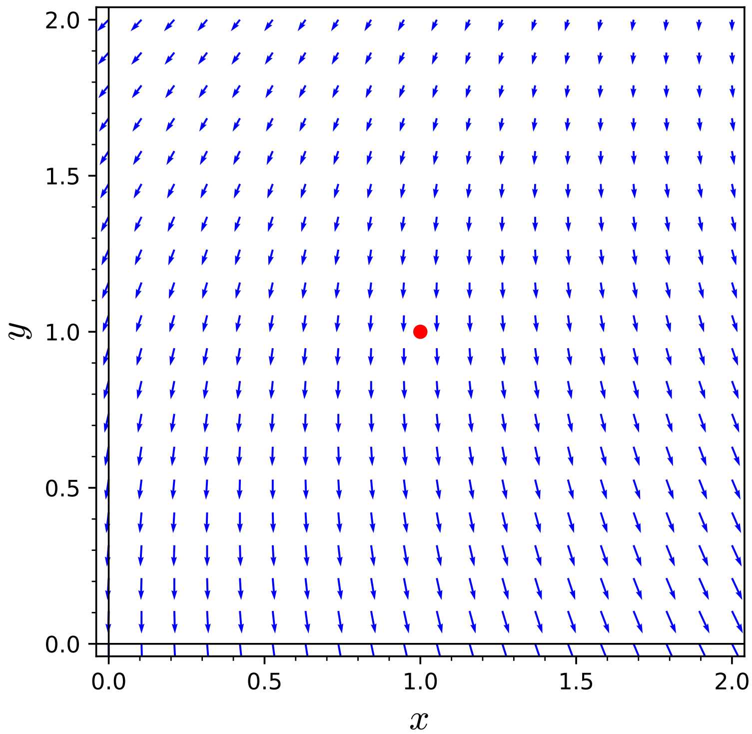


Figure 13.8.19 The trace of $\vec{F}$ in the plane $z = -2$

(c) Next we will look at the vector field in the plane $y = 1$, as displayed in Figure 13.8.20. Do you think a spinner placed on the red point would rotate clockwise, counterclockwise, or not rotate? If the spinner will rotate, you should think about what the axis of rotation would be and whether the rotation should be positive or negative. Summarize your result as a vector representing the rotational strength of your vector field in the plane $y = 1$. Do you think the rotation in this figure is stronger or weaker than in Figure 13.8.19?

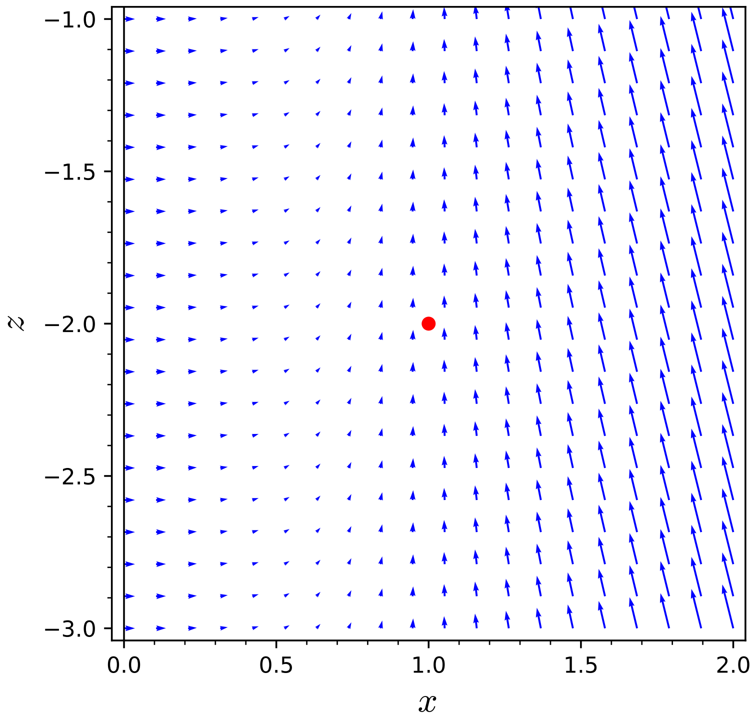


Figure 13.8.20 The trace of $\vec{F}$ in the plane $y = 1$

(d) Finally, consider the trace of $\vec{F}$ in the plane $x = 1$, as displayed in Figure 13.8.21. Do you think a spinner placed on the red point would rotate clockwise, counterclockwise, or not rotate? If the spinner will rotate, you should think about what the axis of rotation would be and whether the rotation should be positive or negative. Summarize your result as a vector representing the rotational strength of your vector field in the plane $x = 1$. How do you think the rotation in this figure compares (i.e., stronger or weaker) to that in Figure 13.8.19 and Figure 13.8.19?

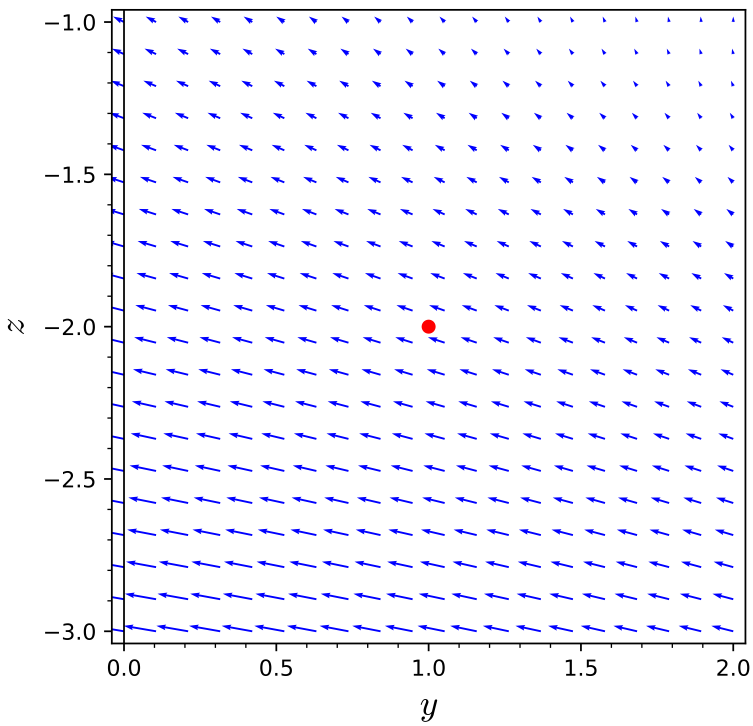


Figure 13.8.21 The trace of $\vec{F}$ in the plane $x = 1$

(e) Summarize your prediction to what you think the three-dimensional rotational strength of the vector field will be at the point $(1, 1, -2)$ in the form of three-dimensional vector.

(f) Compute the curl of $\vec{F} = \langle x - y, y + 2z, x^2 \rangle$. Specifically, what is $\text{curl}(\vec{F})$ at the point $(1, 1, -2)$?

(g) Compare the result of the curl calculation in part f to your prediction from part e. You likely found it difficult to estimate the magnitude, so your answer there may be incorrect. Hopefully, you did get the signs of the components and their relative strengths (i.e., which is biggest) correct. If you did not, go back and review the previous parts and explain why the calculated components match with the rotational strength for each of the three figures.

Activity 13.8.4. In this activity, we will work on calculating curl algebraically and interpreting it.

(a) Consider the vector field $\vec{F} = \langle x, y, z \rangle$. If we plot this vector field in any plane through the origin, we will see the vector field shown in Figure 13.8.23. This two-dimensional vector field has no rotation. The projection of $\operatorname{curl}(\vec{F})(0,0,0)$ onto any direction therefore must give the zero vector. The only vector that has a zero projection in every direction is the zero vector. Verify this geometric argument for the curl of $\vec{F}$ by doing the calculations necessary to show that $\operatorname{curl}(\langle x, y, z \rangle) = \vec{0}$.

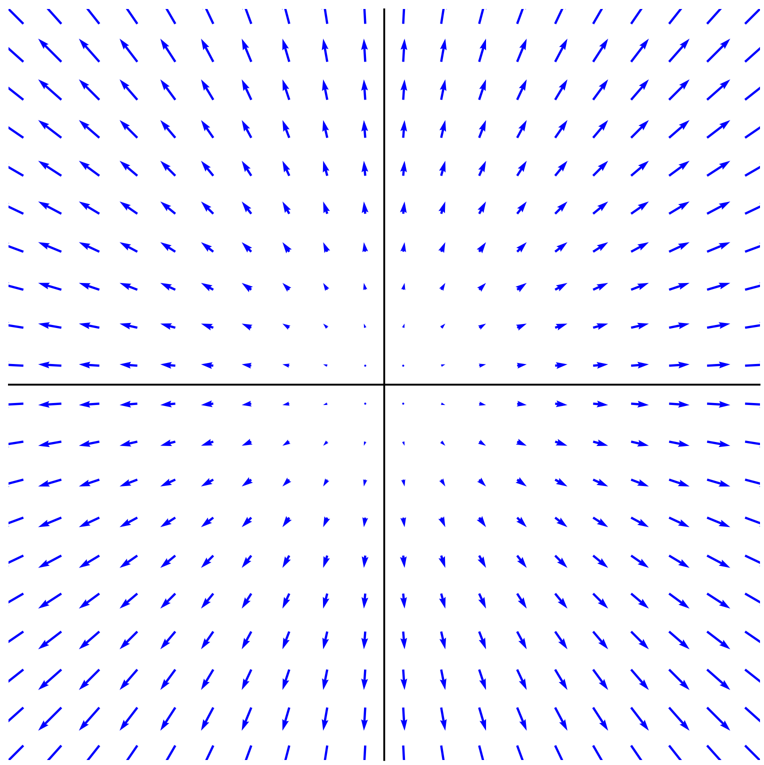
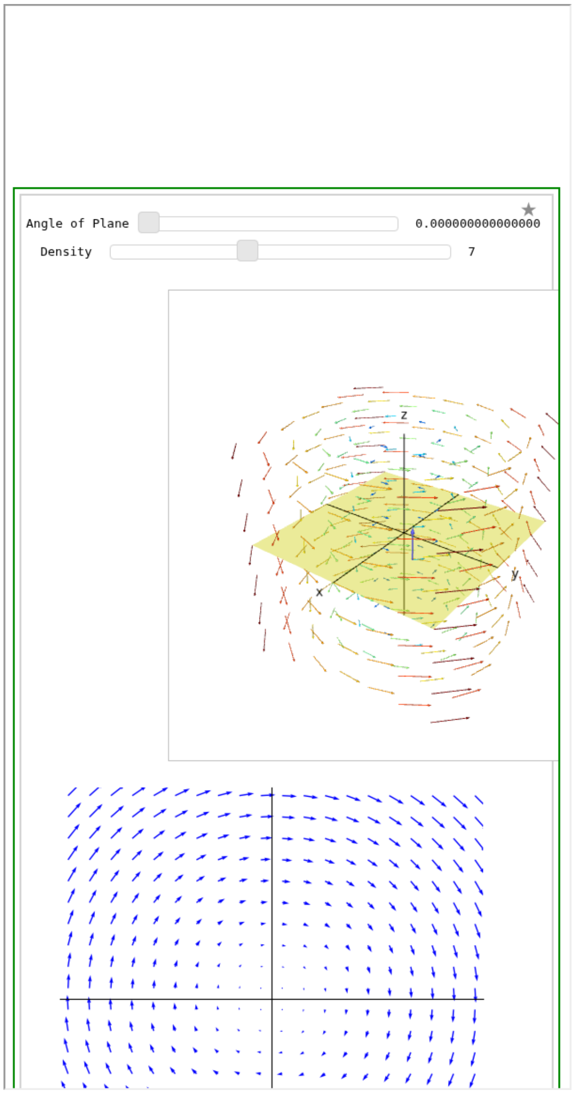


Figure 13.8.23 The vector field $\langle x, y, z \rangle$ on every plane through the origin

(b) Now consider the vector field $\vec{F} = \langle -y, x, 0 \rangle$, which is shown in Figure 13.8.24. The figure includes both an interactive three-dimensional plot of $\vec{F}$ as well as a two-dimensional plot of $\vec{F}$ in the yellow plane, which can be adjusted using the "Angle of Plane" slider.



Standalone

In Context

Embed

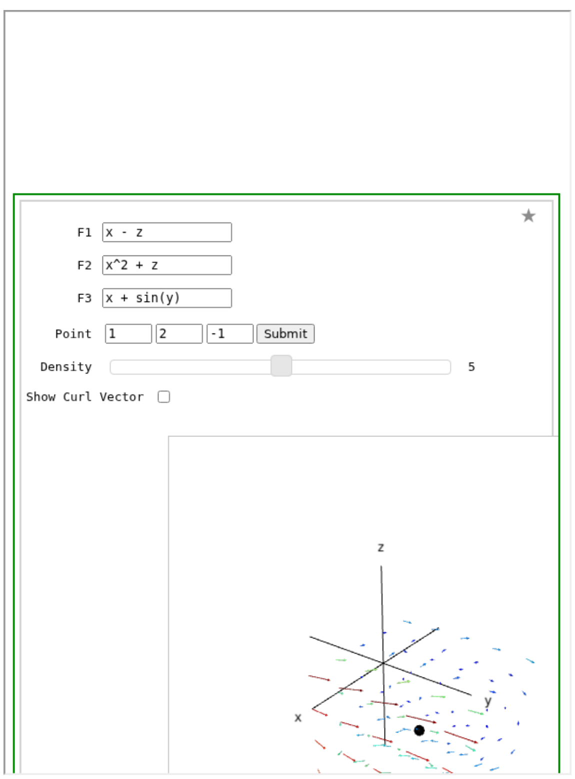
Figure 13.8.24 Two views of the vector field $\langle -y, x, 0\rangle$

(i) Based on the figure, would you estimate the components of $\text{curl}(\vec{F})$ to be positive, negative, or zero at the origin? Does your answer change if you pick a different point?

(ii) Calculate $\text{curl}(\vec{F})$ algebraically. Does the curl of this vector field vary depending on the point at which the curl is measured?

(iii) Explain why the rotational strength of the vector field on a titled plane through the origin will be given by $2\cos(\theta)$, where θ is the angle between $\hat{k}$ and the normal vector of the tilted plane.

(c) In Figure 13.8.25 you can plot a vector field in a region around a point of your choosing in order to look at the rotational properties of the vector field. The check box in Figure 13.8.25 will show the curl vector at the base point specified so you can make sense of your vector field and its curl.



Standalone

In Context

Embed

Figure 13.8.25 A plot of the vector field $\langle F_1, F_2, F_3 \rangle$

Use the figure to estimate the direction of $\operatorname{curl}(\langle x - z, x^2 + z, x + \sin(y) \rangle)$ at the point $(1, 2, -1)$. Confirm your estimate by calculating the curl at this point algebraically. Is there any point at which the direction of greatest rotational strength of this vector field has negative x-component? If there is, find such a point. If not, explain why not.

13.9 Green's Theorem

Preview Activity 13.9.1. We will consider the vector field $\vec{F} = \langle 2y, 3x^2y\rangle$, which is defined on the entire xy-plane. Suppose that we want to calculate the circulation of $\vec{F}$ around the circle C of radius 2, centered at $(0,0)$, and oriented counterclockwise.

(a) Verify that $\vec{F}$ is not path-independent by calculating the circulation of $\vec{F}$ around the circle C (Use Theorem 13.5.3). The SageMath cell below is set up to assist you with this, but you will need to supply a parametrization of C on line 4.

```
var('r,t,x,y')
F = vector([2*y,3*x^2*y])
#Supply a parametrization of C on the next line inside the [ ]
C = vector([ , ])
integral(F(x=C[0],y=C[1]).dot_product(derivative(C,t)),t,0,2*pi)
```

(b) Recall from Section 13.8 that if $\vec{F} = \langle F_1(x,y), F_2(x,y)\rangle$ is a vector field, then the circulation density (in 2D) is given by

$$\text{Circulation Density at the point } (a,b) = \frac{\partial F_2}{\partial x}(a,b) - \frac{\partial F_1}{\partial y}(a,b)$$

What is the circulation density of $\vec{F} = \langle 2y, 3x^2y\rangle$?

(c) Sketch the curve C and shade the region it bounds. Describe the region bounded by C in both rectangular and polar coordinates.

(d) Calculate the double integral of $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$ over the region inside the circle C. This integral is not the most fun to do by hand, so a SageMath cell has been provided to assist you.

```
integral(integral( FUNCTION HERE , var1, start, stop), var2, start, stop)
```

(e) What do you notice about your results to parts (a) and (d)? Do you think this will happen in general? Write a sentence or two about what you think is happening here.

Activity 13.9.2. For each of the curves described below, find the circulation of the given vector field around the curve. Do this both by calculating the line integral directly as well as by calculating the double integral from Green's Theorem.

(a) The curve C_1 is the circle of radius 3 centered at the point $(2,1)$ (oriented counterclockwise) and the vector field is $\vec{F} = \langle y^2, 5x + 2xy \rangle$.

(b) The curve C_2 is the triangle with vertices $(0,0)$, $(3,0)$, and $(3,3)$ (oriented counterclockwise) and the vector field is $\vec{G} = \langle y^2, 3xy \rangle$.

Activity 13.9.3. Consider the vector field $\vec{F} = \frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$. Notice that $\vec{F}$ is smooth everywhere in the plane other than at the point $(0,0)$. This vector field is plotted in Figure 13.9.6, but we have not plotted the vectors close to the origin as their magnitudes get so large that they make it hard to interpret the figure. Here Green's Theorem applies to any simple closed curve C that neither passes through $(0,0)$ nor bounds a region containing $(0,0)$.

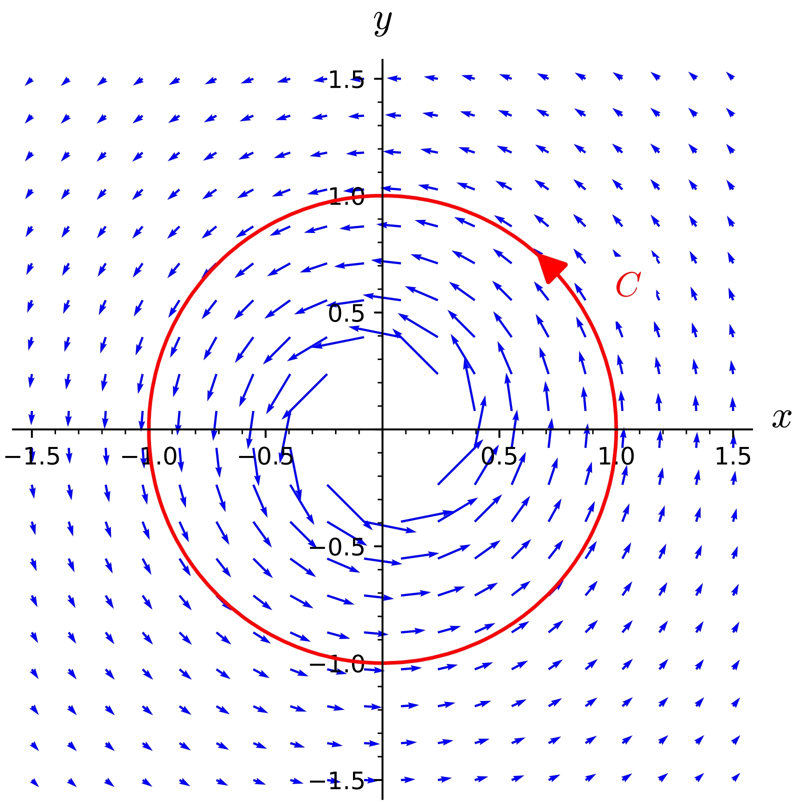


Figure 13.9.6 The vector field $\vec{F}$

(a) Find the circulation density of $\vec{F}$ (i.e., the integrand of the double integral in Green's Theorem).

(b) Suppose that C is the unit circle centered at the origin. Without doing any calculations, what can you say about $\oint_C \vec{F} \cdot d\vec{r}$? What does this tell you about if $\vec{F}$ is path-independent?

(c) What would you get if you integrated the circulation density of $\vec{F}$ over the region bounded by C?

(d) Do the previous two parts contradict Green's Theorem? Explain your reasoning.

(e) Is the vector field $\vec{G} = \dfrac{x}{x^2+y^2}\hat{\imath} + \dfrac{y}{x^2+y^2}\hat{\jmath}$, which is shown in Figure 13.9.7, path-independent? Why or why not?

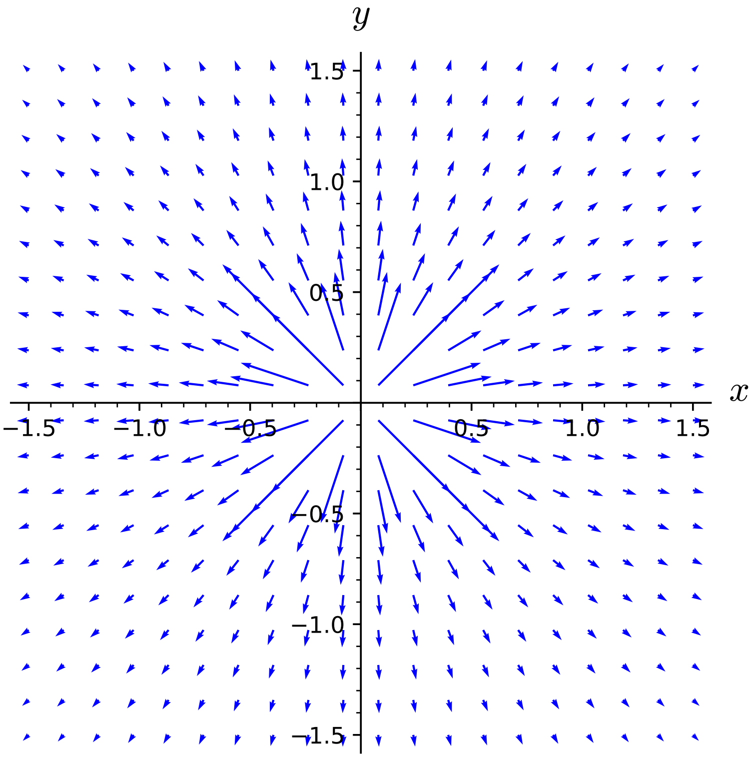


Figure 13.9.7 The vector field $\vec{G}$

(f) Suppose that C is the unit circle centered at the origin. Find $\oint_C \vec{G} \cdot d\vec{r}$. Can you do this using Green's Theorem?

13.10 Parameterizations of Surfaces and Surface Area

Preview Activity 13.10.1. Recall the standard parameterization of the unit circle given by

$$x(t) = \cos(t) \qquad \text{and} \qquad y(t) = \sin(t),$$

where $0 \leq t \leq 2\pi$.

(a) Find a parameterization of a circle of radius of 2 in $\mathbb{R}^3$ centered at $(0, 0, 1)$ and lying in the plane $z = 1$. Be sure to give the bounds on the parameter as well.

(b) Find a parameterization of a circle of radius of 2 in $\mathbb{R}^3$ centered at $(0, 0, -5)$ and lying in the plane $z = -5$. Be sure to give the bounds on the parameter as well.

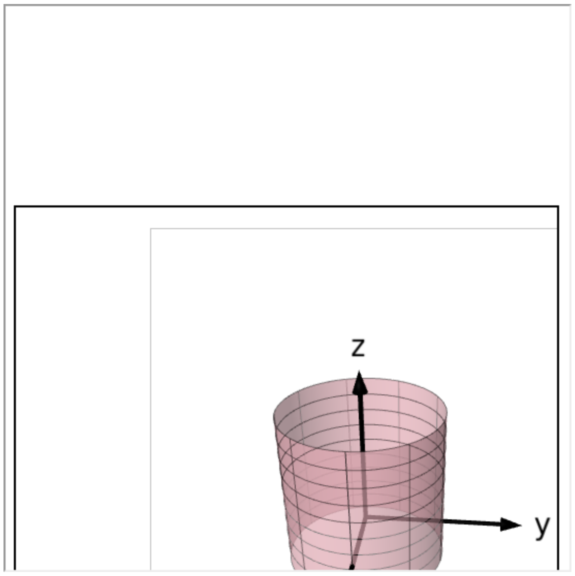
(c) Find a parameterization of a circle of radius of 2 in $\mathbb{R}^3$ centered at $(0, 0, a)$ and lying in the plane $z = a$. Be sure to give the bounds on the parameter as well.

(d) A right circular cylinder surface centered on the z-axis, such as the one shown in Figure 13.10.1, can be thought of as being built from a stack of circles of fixed radius centered at points along the z-axis. Find a parameterization of a right circular cylinder surface of radius 2, centered on the z-axis with heights from b to c. In other words, express the x-, y-, and z-coordinates of points on the right circular cylinder in terms of two variables. Use the parameter s to measure the height of the circle above or below the xy-plane and use the parameter t to measure the location around the circle centered on the z-axis. Use the previous part to write out each of the following:

$$x(s,t) = \boxed{} \qquad y(s,t) = \boxed{} \qquad z(s,t) = \boxed{}$$

with the bounds on s and t given by

$$\boxed{} \leq s \leq \boxed{} \quad \text{and} \quad \boxed{} \leq t \leq \boxed{}$$



Standalone

In Context

Embed

Figure 13.10.1 A right circular cylinder centered on the z-axis

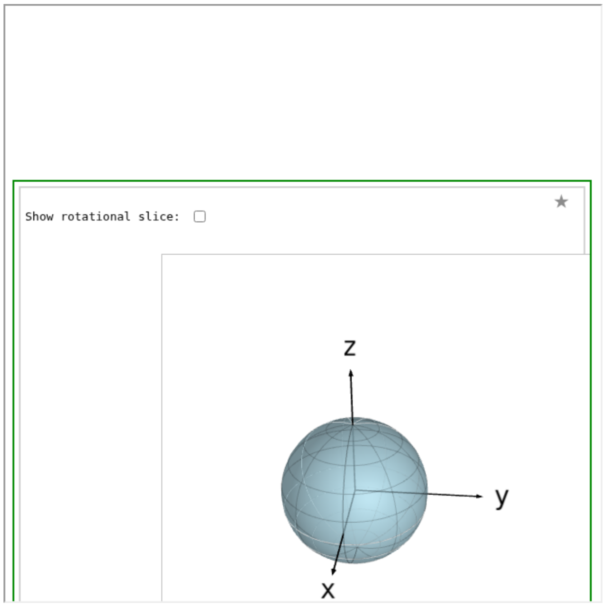
(e) Just as a right circular cylinder can be viewed as a "stack" of circles of fixed radius, a cone can be viewed as a stack of circles with *varying* radius. In particular, the cone $z^2 = x^2 + y^2$ has contours that are circles centered on the z-axis with radius changing according to the rule $r = z$. Find a parameterization of the cone $z^2 = x^2 + y^2$ with heights from b to c. In other words, we want to express the x, y, and z coordinates of points on the cone in terms of two variables. We will let the parameter s measure the height of the circle (in our stack) above or below the xy-plane and let the parameter t measure the location around the circle centered on the z-axis. Write out each of the following:

$$x(s,t) = \qquad y(s,t) = \qquad z(s,t) =$$

with the bounds on s and t given by

$$\leq s \leq \qquad \text{and} \qquad \leq t \leq$$

Activity 13.10.2. The goal of this activity is to find a parametrization of the sphere of radius R centered at the origin, as shown in Figure 13.10.6. Check the box at the top of the plot to highlight what a semicircular slice of the sphere looks like along a slice with a constant θ coordinate. Notice that this slice shows how the sphere can be made by rotating this half circle around the z-axis.



Standalone

In Context

Embed

Figure 13.10.6 A plot of a sphere of radius R centered at the origin

(a) One approach to parameterizing the sphere of radius R is similar to the approach to parameterizing the torus as a surface of revolution around the z-axis. In other words, let the parameter t represent the location as a rotation around the z-axis. This allows for parameterization the sphere as

$$x(s,t) = r(s)\cos(t), \quad y(s,t) = r(s)\sin(t), \quad z(s,t) = z(s), \tag{13.10.1}$$

where $r(s)$ and $z(s)$ describe the other cylindrical coordinates in terms of another parameter.

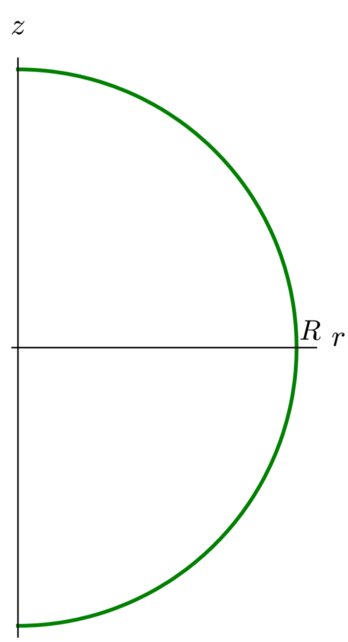


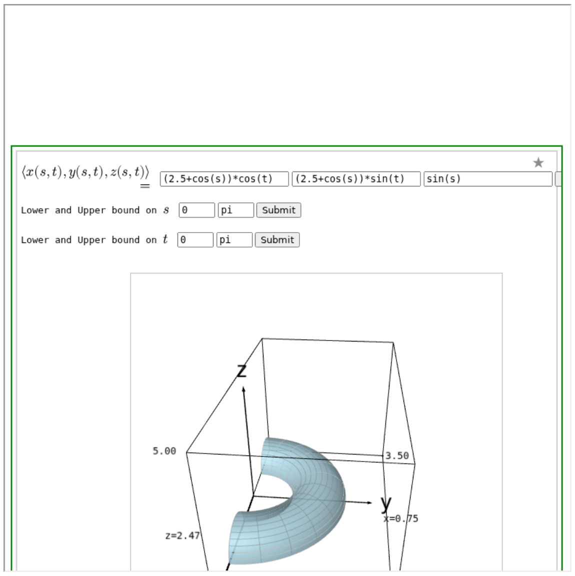
Figure 13.10.7 A plot of a half circle in the rz-plane representing the slice of the sphere rotated around the z-axis

We want to describe the points on the green half circle shown in Figure 13.10.7 in terms of the parameter s. Remember that there is not a unique way to parameterize a curve. Give a parameterization of the green half circle in Figure 13.10.7 as a function of the parameter s. Be sure to state the bounds on s.

(b) Combine your answer to the previous part with (13.10.1) to create a parameterization of the sphere of radius R. Substitute your answer for the previous part into (13.10.1), writing your answer as parametric functions for x, y, and z. Be sure to state the bounds on the parameters s and t.

$$x(s,t) = \square$$
$$y(s,t) = \square$$
$$z(s,t) = \square$$
$$\square \leq s \leq \square$$
$$\square \leq t \leq \square$$

(c) Use Figure 13.10.8 to verify that your parameterization for the previous part plots the sphere of radius 3 centered at the origin. Remember to adjust the upper and lower bounds of s and t to match your parameterization.



Standalone

In Context

Embed

Figure 13.10.8 An interactive plot where readers can input their parameterization and bounds

(d) Next we use spherical coordinates to create a parameterization of the sphere of radius R. We want to express the rectangular coordinate equation $x^2+y^2+z^2 = R^2$ in spherical coordinates. State an equation in spherical coordinates that describes the same set of points as $x^2 + y^2 + z^2 = R^2$.

(e) Surfaces of the form $z = f(x, y)$ were easily parameterized because each of the coordinates can be expressed in terms of two variables: x and y. We can apply a similar idea here because in spherical coordinates, the sphere has one spherical coordinate fixed and the other two spherical coordinates can be used as parameters. Write a couple of sentences to describe which spherical coordinate s will act as and which coordinate t will act as. Include bounds on these parameters as part of your descriptions.

(f) To create a spherical coordinate parameterization of $x^2 + y^2 + z^2 = R^2$, substitute the constant value for the appropriate spherical coordinate and s or t, as determined by your choices in the previous part, for the other two spherical coordinates into the following transformation equations:

$$x = \rho\cos(\theta)\sin(\phi) \qquad y = \rho\sin(\theta)\sin(\phi) \qquad z = \rho\cos(\phi)$$

This gives equations for x, y, and z in terms of the parameters s and t, as well as bounds on both s and t:

$$x(s,t) = \underline{\qquad} \qquad y(s,t) = \underline{\qquad} \qquad z(s,t) = \underline{\qquad}$$

with the bounds on s and t given by

$$\underline{\qquad} \leq s \leq \underline{\qquad} \text{ and } \underline{\qquad} \leq t \leq \underline{\qquad}.$$

(g) Use Figure 13.10.8 to verify that your parameterization does produce the sphere of radius 3.

Activity 13.10.3. This activity computes the surface area of a right circular cylinder. In particular, consider the cylinder with radius a and height h defined parametrically by

$$\vec{r}(s,t) = a\cos(s)\hat{\imath} + a\sin(s)\hat{\jmath} + t\hat{k}$$

for $0 \leq s \leq 2\pi$ and $0 \leq t \leq h$, as shown in Figure 13.10.15.

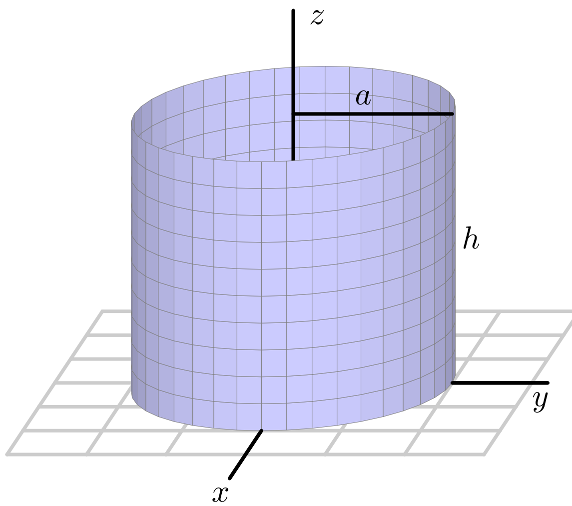


Figure 13.10.15 A part of the cylinder of radius a about the z-axis

(a) Calculate $\vec{r}_s, \vec{r}_t, \|\vec{r}_s \times \vec{r}_t\|$ based on the parameterization given above.

(b) Use the calculations from the previous part to set up an iterated integral to determine the surface area of this cylinder.

(c) Evaluate your iterated integral from the previous part.

(d) One way to think about the surface area of a cylinder is to cut the cylinder horizontally and find the perimeter of the resulting cross sectional circle, then multiply by the height. Calculate the surface area of the given cylinder using this alternate approach, and compare your result the value from the previous part.

Activity 13.10.4. Let $z = f(x, y)$ define a smooth surface, and consider the parameterization $\vec{r}(s,t) = \langle s, t, f(s,t) \rangle$.

(a) Let D be a region in the domain of f. Using equation (13.10.2), show that the area of the surface defined by the graph of f over D is

$$\iint_D \sqrt{(f_x(x,y))^2 + (f_y(x,y))^2 + 1}\; dA.$$

(b) Use the formula developed in the previous part to calculate the area of the surface defined by $f(x, y) = \sqrt{4 - x^2}$ over the rectangle $D = [-2, 2] \times [0, 3]$.

(c) Observe that the surface of the solid describe the previous part is half of a circular cylinder. Use the standard formula for the surface area of a cylinder to calculate the surface area in a different way, and compare your result from above.

13.11 Flux Integrals

Preview Activity 13.11.1. In this preview activity, we will explore the parameterizations of a few familiar surfaces and confirm some of the geometric properties described in the introduction above.

(a) Use the ideas from Section 13.10 to give a parameterization $\vec{r}(s,t)$ of the following surface. Be sure to specify the bounds on each of your parameters.

S_1: the right circular cylinder centered on the x-axis of radius 2 when $0 \leq x \leq 5$

(b) Draw a graph of S_1 from the previous part. Label the points that correspond to (s,t) points of $(0,0)$, $(0,1)$, $(1,0)$, and $(2,3)$.

(c) For the parameterization from part a, calculate $\vec{r}_s$, $\vec{r}_t$, and $\vec{r}_s \times \vec{r}_t$.

(d) For the parameterization from part a, find the value for $\vec{r}_s$,$\vec{r}_t$, and $\vec{r}_s \times \vec{r}_t$ at the (s,t) points of $(0,0)$, $(0,1)$, $(1,0)$, and $(2,3)$.

(e) Draw your vector results from d on your graph and confirm the geometric properties described in the introduction to this section. Namely, $\vec{r}_s$ and $\vec{r}_t$ should be tangent to the surface, while $\vec{r}_s \times \vec{r}_t$ should be orthogonal to the surface (in addition to $\vec{r}_s$ and $\vec{r}_t$).

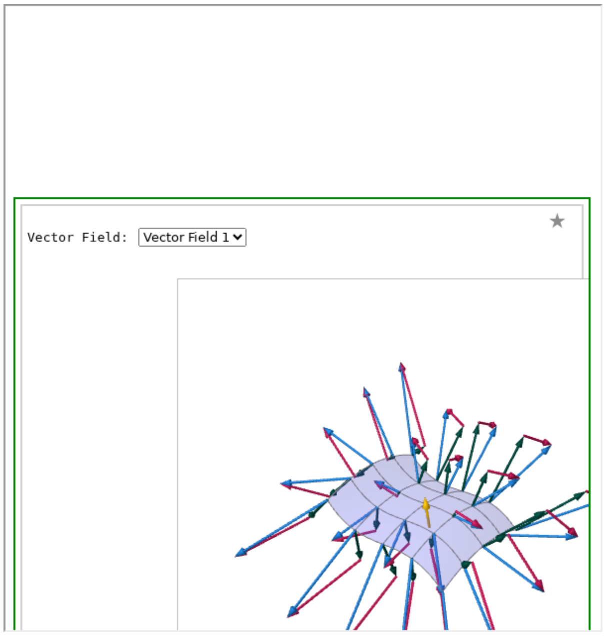
(f) Repeat steps (a) through (e) for the following two surfaces:

(i) S_2: the sphere centered at the origin of radius 3

(ii) S_3: the first octant portion of the plane $x + 2y + 3z = 6$

Activity 13.11.2 Visualizing flux through a surface.. In this activity, you will compare the net flow of different vector fields through our sample surface. In Figure 13.11.5 you can select between five different vector fields. Once you select a vector field, the vector field for a set of points on the surface will be plotted in blue. Each blue vector will also be split into its normal component (in green) and its tangential component (in magenta). The yellow vector defines the direction for positive flow through the surface.

Look at each vector field and order the vector fields from greatest net flow through the surface to least net flow through the surface. Remember that a negative net flow through the surface should be lower in your rankings than any positive net flow.



Standalone

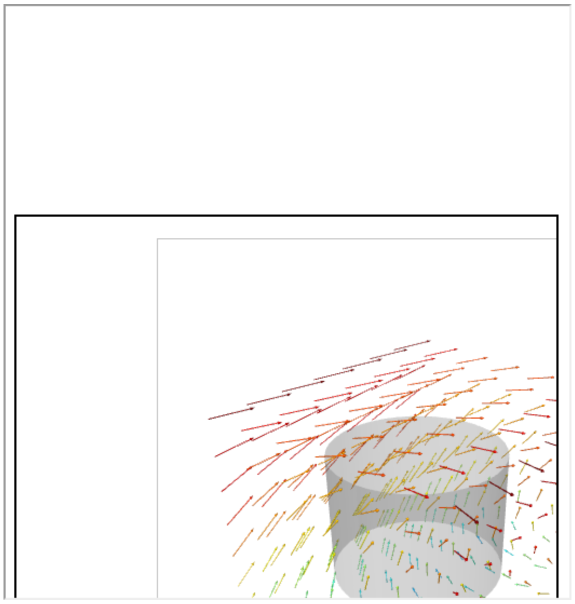
In Context

Embed

Figure 13.11.5 Five vector fields flowing through a surface, with decomposition into normal and tangential components

Activity 13.11.3 Checking the Visualization for Flux..

(a) Figure 13.11.12 shows a plot of the vector field $\vec{F} = \langle y, z, 2 + \sin(x) \rangle$ and a right circular cylinder of radius 2 and height 3 (with open top and bottom). Consider the vector field going into the cylinder (toward the z-axis) as corresponding to positive flux.



Standalone

In Context

Embed

Figure 13.11.12 The vector field $\vec{F} = \langle y, z, 2 + \sin(x) \rangle$ and a right circular cylinder

(i) Reasoning graphically, do you think the flux of $\vec{F}$ throught the cylinder will be positive, negative, or zero? Write a few sentences jutifying your answer.

(ii) Parameterize the right circular cylinder of radius 2, centered on the z-axis, for $0 \leq z \leq 3$. Be sure to give bounds on your parameters.

(iii) Based on your parameterization, compute $\vec{r}_s$, $\vec{r}_t$, and $\vec{r}_s \times \vec{r}_t$. Confirm that these vectors are either orthogonal or tangent to the right circular cylinder. Is your orthogonal vector pointing in the direction of positive flux or negative flux?

(iv) Use your parametrization to write $\vec{F}$ as a function of s and t.

(v) Compute the flux of $\vec{F}$ through the parameterized portion of the right circular cylinder.

(vi) Does your computed value for the flux match your prediction from earlier?

(vii) Use Figure 13.11.13 to make an argument about why the flux of $\vec{F} = \langle y, z, 2 + \sin(x) \rangle$ through the right circular cylinder is zero.

Standalone

In Context

Embed

Figure 13.11.13 The vector field $\vec{F} = \langle y, z, 2 + \sin(x) \rangle$ with normal and tangential components plotted on a right circular cylinder

(b) Write a couple of sentences to explain how the results of the flux calculations would be different if we used the vector field $\vec{F} = \left\langle y, z, \cos(xy) + \frac{9}{z^2+6.2} \right\rangle$ and the same right circular cylinder.

(c) Write a couple of sentences to explain how the results of the flux calculations would be different if we used the vector field $\vec{F} = \langle y, -x, 3 \rangle$ and the same right circular cylinder.

13.12 Surface Integrals of Scalar Valued Functions

Preview Activity 13.12.1. In Preview Activity 13.6.1 we looked at how to understand line integrals of scalar functions through the analogy of running a mining machine along a given path. In short, the amount of copium mined depended on the density of copium at points on the path and the length of the path driven by the mining rig.

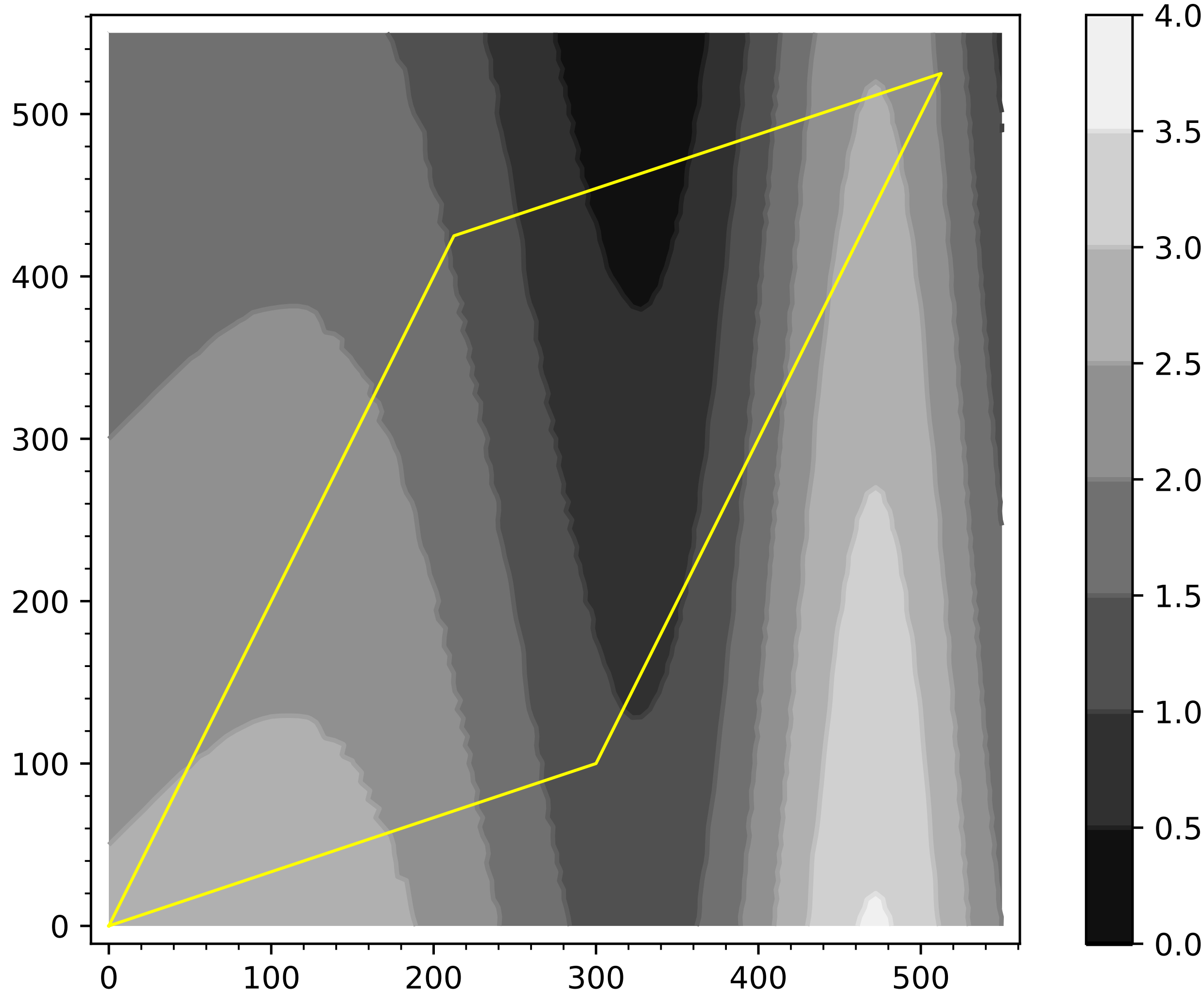


Figure 13.12.1 A plot of land with density of copium deposits and the edges of the plot drawn in yellow

The opening tasks of Preview Activity 13.6.1 had you estimating the amount of copium mined by driving along the edge of a plot of land (drawn in yellow on Figure 13.12.1). This interpretation meant that the scalar function we were using measured the linear density of copium. Thus, the Riemann sum we computed was the product of linear density of copium with the distance traveled.

(a) In this task, we will interpret Figure 13.12.1 as a contour graph of the density of copium per unit area. This will allow us to compute the total amount of copium in our mining area by setting up a double integral. To approximate the total amount of copium available in our mine, do the following:

(i) Break the mining plot (the area inside the yellow segments of Figure 13.12.1) into three pieces. Estimate the area of the three pieces you are using. Write a few sentences explaining your methods of estimating the areas.

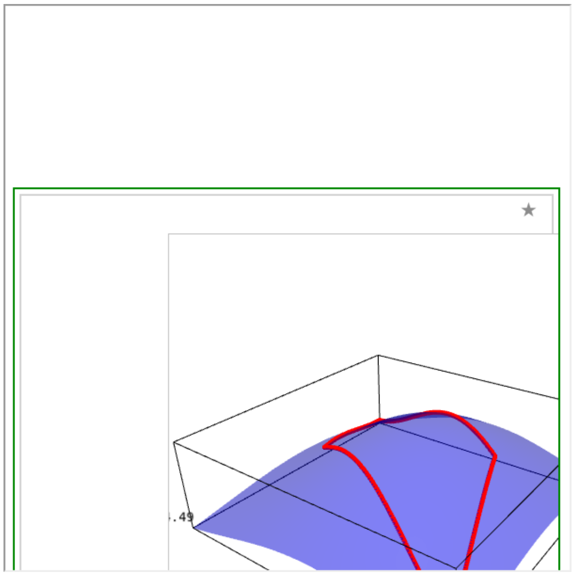
(ii) For each of the three pieces you used in part a.i, estimate the average density of copium on the piece. Write a few sentences explaining your methods of estimating the average density on each piece.

(iii) Give an estimate for the total amount of copium on the mining plot and explain your computation.

(b) What if instead of your mining plot being on a flat piece of land as represented in Figure 13.12.1, your mining plot was on a hill as represented in Figure 13.12.2. If we had the same copium density plot as a function of the (x, y) coordinates, which of the following would you expect to be true?

- the total amount of copium available on Figure 13.12.2 is greater than on Figure 13.12.1
- the total amount of copium available on Figure 13.12.2 is lesser than on Figure 13.12.1
- the total amount of copium available on Figure 13.12.2 is the same as on Figure 13.12.1

Explain your reasoning for your choice.



Standalone

In Context

Embed

Figure 13.12.2 A 3D plot of the region including our mining area (shown with boundary in red)

Activity 13.12.2. In this activity, we will try to understand the scalar surface integral by looking at whether the value of the scalar surface integral will be positive, negative, or zero over common surfaces. In each part below, you are given a function and a surface. For each surface, first draw a plot of the surface and make sure you have labeled a proper scale for each coordinate direction. Then reason if the given surface integral is positive, negative, or zero. Be sure to justify your answers in terms of the function being integrated *and* the particulars of the surface of integration.

(a) For S_1 defined as the top half ($z \geq 0$) of the sphere of radius one centered at the origin, consider the surface integral $\iint_{S_1} x\, dS$.

(b) For S_2 defined as the bottom half ($z \leq 0$) of the sphere of radius one centered at the origin, consider the surface integral $\iint_{S_2} z\, dS$.

(c) For S_3 defined as the disc of radius one centered at $(1, 0, 0)$ on the plane $x = 1$, consider the surface integral $\iint_{S_3} x + z\, dS$.

(d) For S_1 as defined above, consider the surface integral $\iint_{S_1} x + z\, dS$.

Activity 13.12.3 Explaining Properties of Scalar Surface Integrals.. In this activity, we will be explaining each of the properties from Properties of Scalar Line Integrals in the context of a new analogy. We have just purchased a plot of land that spans two mountains, Sugar Mountain and Spice Mountain. We will label the plot of land on Sugar Mountain S_1 and the plot of land on Spice Mountain S_2. Unfortunately there are two types of toxic organisms on the surface of your new land, which may explain why you paid so little for the land. Let f be the density of the toxic fungus on your new plot of land and let g be the density of toxic bacteria on the new plot.

(a) Explain in your own words what $\iint_{S_1} f\,dS$ means in the above analogy and what exactly would be measured by this scalar surface integral.

(b) Explain in your own words what $\displaystyle\iint_{S_1} (2f)\,dS = 2\iint_{S_1} f\,dS$ means in the new analogy. It may be helpful to describe each side of the equation separately and say why they are equal in the analogy.

(c) Explain in your own words what $\displaystyle\iint_{S_2} (f+g)\,dS = \iint_{S_2} f\,dS + \iint_{S_2} g\,dS$ means in the new analogy. It may be helpful to describe each side of the equation separately and say why they are equal in the analogy.

(d) Explain in your own words what $\displaystyle\iint_{-S_2} f\,dS = \iint_{S_2} f\,dS$ means in the new analogy. It may be helpful to describe each side of the equation separately and say why they are equal in the analogy.

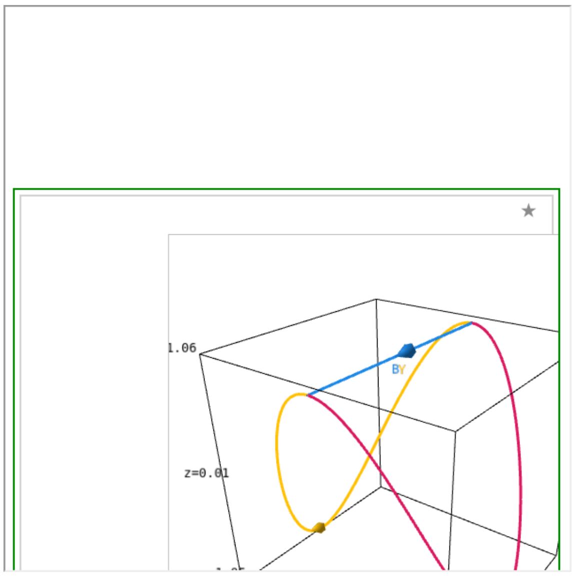
(e) Explain in your own words what $\iint_{S_1+S_2} f\,dS = \iint_{S_1} f\,dS + \iint_{S_2} f\,dS$ means in the new analogy. It may be helpful to describe each side of the equation separately and say why they are equal in the analogy.

Activity 13.12.4. Compute the value of the scalar surface integral in part c of Subsection 13.12.2. That is, compute $\iint_{S_3} x + z\, dS$ where S_3 is the disc of radius one centered at $(1, 0, 0)$ on the plane $x = 1$. Explain why your answer makes sense geometrically.

13.13 Stokes' Theorem

Preview Activity 13.13.1. In this activity, we will look at how we can apply the ideas about circulation along overlapping curves from the beginning of Subsection 13.9.2 to curves in space.

(a) For this part, consider the curves in Figure 13.13.1, where the yellow curve is Y, the blue curve is B, and the magenta curve is M.



Standalone

In Context

Embed

Figure 13.13.1 Three curves in space

You should also go back and refamiliarize yourself with our notation for combining paths (as used in line integrals) from Convention 13.3.13. In our convention, $Y + M$ would be closed loop, but $Y - M$ would not make sense because the segment $-M$ does not begin where Y begins.

(i) Using the three segments in Figure 13.13.1, write out at least four different closed curves in terms of B, Y, and M. (Remember to consider orientation!)

(ii) Let $C_1 = M + B$ and $C_2 = Y - B$. Describe the curve given by $C_1 + C_2$.

(iii) Write a couple of sentences explaining how the circulation around $C_1 + C_2$ would compare to the circulation around C_1 and the circulation around C_2. Write an equation in terms of $\int_{C_1} \vec{F} \cdot d\vec{r}$, $\int_{C_2} \vec{F} \cdot d\vec{r}$, and $\int_{C_1+C_2} \vec{F} \cdot d\vec{r}$.

(iv) Explain how your arguments or equations from any of the parts above would or would not change if you considered the curves depicted in Figure 13.13.2.

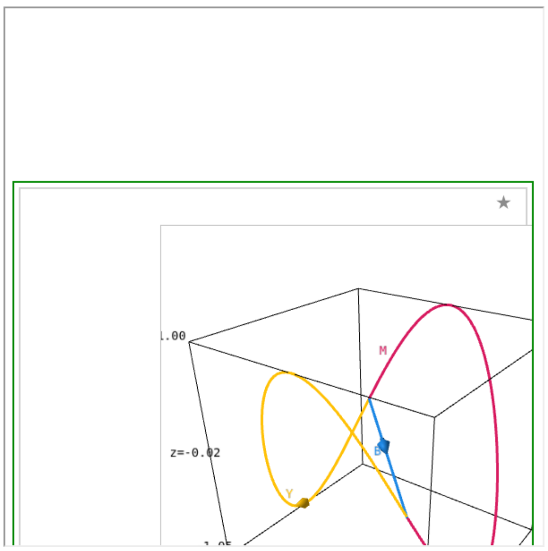


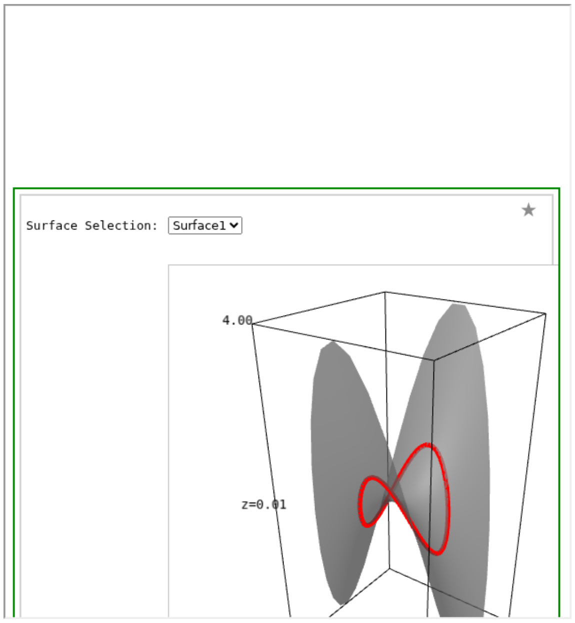
Figure 13.13.2 Three slightly different curves in space

Standalone

In Context

Embed

(b) Let C be the simple closed curve consisting of the yellow and magenta curves in Figure 13.13.1. You can see C plotted in red in Figure 13.13.3. The drop-down allows you to select three different surfaces. You can visually verify that each of the three surfaces contains C. Notice that the scale on the z-axis changes as you select different surfaces.



Standalone

In Context

Embed

Figure 13.13.3 Surfaces containing a common simple closed curve

The simple closed curve consisting of the yellow and magenta curves in Figure 13.13.1 can be parameterized by $\langle \cos(t), \sin(t), \cos(2t) \rangle$ with $0 \leq t \leq 2\pi$. Let $C_3 = Y + M$. Use the given parameterization of C_3 to show that C_3 is on each of the following surfaces:

- $x^2 - y^2 = z$
- $z = x^4 - y^4$
- $z = 1 - 2y^2$
- $z = -\cos(\pi\sqrt{x^2+y^2})(x^2 - y^2)$

Activity 13.13.2. In this activity, we will verify Stokes' Theorem by calculating both a line integral and a flux integral.

(a) Consider the vector field $\vec{F} = \langle x^2, y^2, z^2 \rangle$ and the circle C_1 parameterized as $\vec{r}(t) = \langle \sqrt{2}\cos(t), \sqrt{2}\cos(t), 2\sin(t) \rangle$ for $0 \le t \le 2\pi$.

(i) Calculate $\oint_{C_1} \vec{F} \cdot d\vec{r}$ directly using the given parametrization.

(ii) Let S_1 be the hemisphere of the sphere of radius 2 centered at the origin with $y \le x$. Calculate the flux of $\operatorname{curl}(\vec{F})$ through S_1.

(iii) What could you have observed about $\vec{F}$ that would have gotten you the same answer without doing either of the above calculations?

(b) Consider the vector field $\vec{G} = x\hat{\imath} + y^2 z\hat{\jmath} + x^2\hat{k}$ and the curve C_2, which is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ with orientation corresponding to the order the points are listed here.

(i) Find the circulation of $\vec{G}$ along C_2 by calculating the appropriate line integrals.

(ii) The vertices of C_2 lie in a plane. Let S_2 be the portion of this plane lying in the first octant, i.e., the portion with $x, y, z \ge 0$. Find the flux of $\operatorname{curl}(\vec{G})$ through S_2.

(iii) Write a sentence to explain why the sign of your answer to the previous two parts makes sense.

Activity 13.13.3.

(a) Find the circulation of $\vec{F} = \langle 3yz, xz, -xy \rangle$ along the curve C consisting of (given in order of the orientation) the quarter-circle of radius 1 centered at $(0, -2, 0)$ in the plane $y = -2$ from $(0, -2, 1)$ to $(1, -2, 0)$, the line segment from $(1, -2, 0)$ to $(1, 5, 0)$, the quarter-circle of radius 1 centered at $(0, 5, 0)$ in the plane $y = 5$ from $(1, 5, 0)$ to $(0, 5, 1)$, and the line segment from $(0, 5, 1)$ to $(0, -2, 1)$.

(b) Find the circulation of $\vec{G} = 3z^2\hat{\imath} - (z^2 + 2x)\hat{\jmath} + zy\hat{k}$ along the circle in the xy-plane of radius 3 centered at the origin. Assume the counterclockwise orientation of the circle.

Activity 13.13.4. Because Stokes' Theorem requires us to consider a surface (with normal vector) and the boundary of the surface, this activity will give you a chance to practice identifying the boundary of some surfaces in $\mathbb{R}^3$. For each surface below:

i Describe the boundary in words.

ii Find a parametrization for the boundary.

iii Ensure that a person walking along the boundary in the direction of your parametrization with head pointing in the direction of the surface's normal vector would hold their left hand over the surface.

(a) The surface S_1 is the portion of the sphere $x^2+y^2+z^2=4$ with $z \geq x$. Assume the outward orientation on the sphere.

(b) The surface S_2 is the portion of the sphere $x^2+y^2+z^2=4$ with $z \geq 0$. Assume the outward orientation on the sphere.

(c) The surface S_3 is the portion of the hyperbolic paraboloid $z=x^2-y^2$ with $x^2+y^2 \leq 1$. Assume the "upward" orientation, e.g., the normal vector at $(0,0,0)$ is $\hat{k}$.

(d) The surface S_4 is the portion of the cylinder $x^2+y^2=4$ for which $-2 \leq z \leq 2$, assuming the outward orientation.

Activity 13.13.5. In some sense, this activity considers the reverse problem of that considered in Activity 13.13.4. Here, each part of the activity gives you an oriented simple closed curve C in $\mathbb{R}^3$, and your task is to find

- a surface S so that C is the boundary of S and
- a normal vector for the S so that a person walking along C in the direction of the given orientation with head pointing in the direction of your chosen normal vector would have their left hand over S.

You are encouraged to think about multiple possible answers, since as we saw in Preview Activity 13.13.1, there may be more than one reasonable choice of a surface with a particular boundary.

(a) The curve C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ with orientation corresponding to the order the points are listed here.

(b) The curve C is the circle parameterized as $\vec{r}(t) = \langle \sqrt{2}\cos(t), \sqrt{2}\cos(t), 2\sin(t)\rangle$ for $0 \leq t \leq 2\pi$.

(c) The curve C consists of (given in order of the orientation)

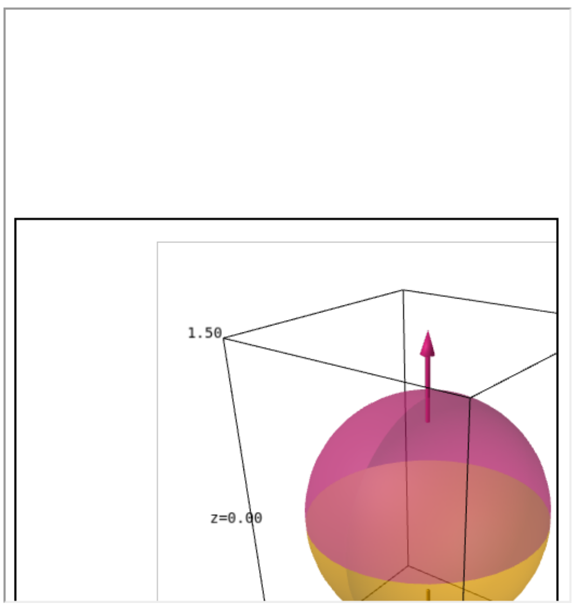
- the quarter-circle $C_1$of radius 2 centered at the origin in the xy-plane from $(2, 0, 0)$ to $(0, 2, 0)$,
- the line segment C_2 from $(0, 2, 0)$ to $(0, 2, 2)$,
- the quarter-circle C_3 of radius 2 centered at $(0, 0, 2)$ in the plane $z = 2$ from $(0, 2, 2)$ to $(2, 0, 2)$, and
- the line segment C_4 from $(2, 0, 2)$ to $(2, 0, 0)$.

13.14 The Divergence Theorem

Preview Activity 13.14.1 Locating sources of a poisonous gas.. We will use three different surfaces to examine the flux through closed surfaces. Let S_{top} be the top half of the unit sphere centered at the origin (graphed in magenta in the figures below). Let S_{bottom} be the bottom half of the unit sphere centered at the origin (graphed in yellow). Finally, let S_{mid} be the unit disk centered at the origin in the xy-plane (graphed in blue). With these definitions, S_{top} and S_{bottom} will make a closed surface given by the unit sphere. The surfaces S_{top} and S_{mid} will enclose the top half of the unit ball, while S_{bottom} and S_{mid} will enclose the bottom half of the unit ball.

In this problem we will be using the surfaces defined above and the flux integrals of a poisonous gas through these surfaces to try to determine whether different regions of space are emitting or absorbing the poisonous gas.

(a) In this part, we will consider the unit ball shown in Figure 13.14.4, with boundary and normal vectors as shown in the plot. If the flux integral of a poisonous gas through S_{top} is 15 and the flux integral of the poison gas through S_{bottom} is -3, is the interior of the sphere emitting or absorbing poisonous gas? Explain your reasoning.



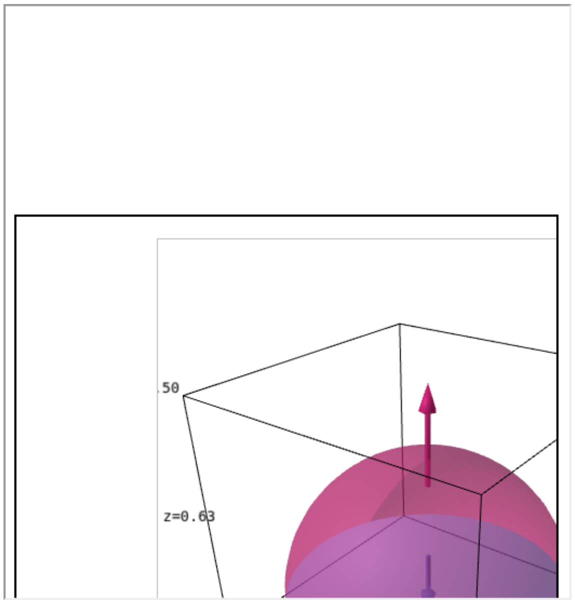
Standalone

In Context

Embed

Figure 13.14.4 Unit ball with boundary given by the combination of S_{top} and S_{bottom}

(b) In this part, we will consider the top half of the unit ball shown in Figure 13.14.5, with boundary and normal vectors as shown in the plot. If the flux integral of a poisonous gas through S_{top} is 15 and the flux integral of the poison gas through S_{mid} is -20, is the top half of the unit ball emitting or absorbing poisonous gas? Explain your reasoning.



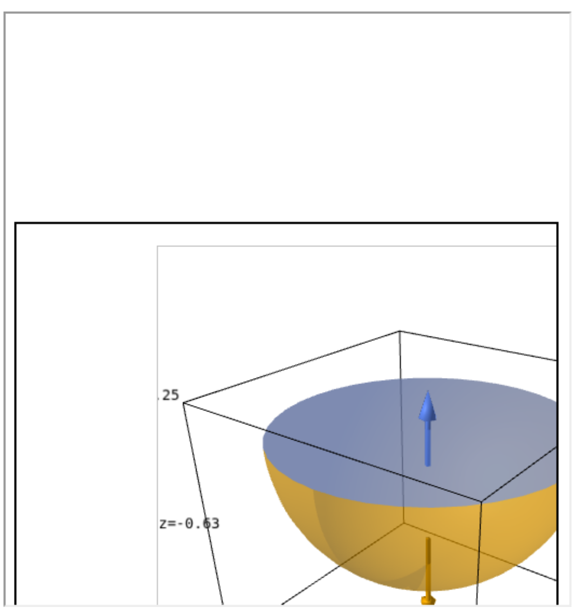
Standalone

In Context

Embed

Figure 13.14.5 Upper half of the unit ball with boundaries given by S_{top} and S_{mid}

(c) In this part, we will consider the bottom half of the unit ball shown in Figure 13.14.6, with boundary and normal vectors as shown in the plot. Based on the information given in the previous two parts, what will the flux integrals of the poison gas be for S_{bottom} and S_{mid} be in this case? Be sure to pay attention to the orientation of what we consider positive flow. Explain your reasoning.



Standalone

In Context

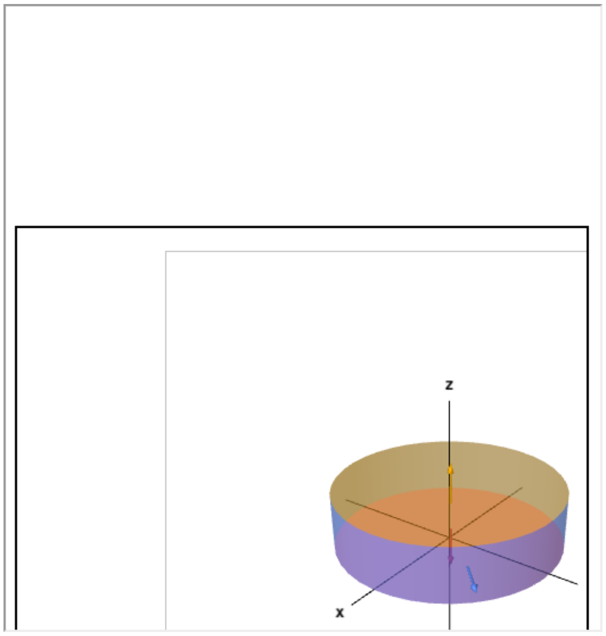
Embed

Figure 13.14.6 Lower half of the unit ball with boundaries given by S_{mid} and S_{bottom}

(d) Using your answer from the previous part, is the bottom half of the unit ball emitting or absorbing poisonous gas? Explain your reasoning.

Activity 13.14.2. In this activity, we will look at calculating both sides of a non-trivial example of the Divergence Theorem. We will look at the region inside the right circular cylinder shown in Figure 13.14.10. Let S be the closed surface formed by combining S_{top} (in yellow), S_{sides} (in blue), and S_{bottom} (in magenta). The solid volume Q is the volume bounded by S. The region shown has radius 2, and its height is 1. The vector field we consider in this activity is given by

$$\vec{F} = \langle xy - 2z, y^2 - yz, 3x + z^2 \rangle.$$



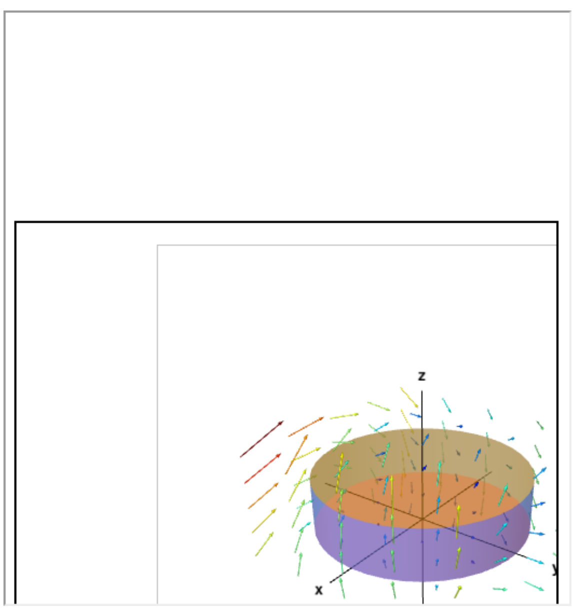
Standalone

In Context

Embed

Figure 13.14.10 A closed cylindrical surface

(a) Figure 13.14.11 shows the vector field $\vec{F}$ on a region around S. Without doing any computations, write a couple of sentences to explain if you think the flux of $\vec{F}$ through S will be positive, negative, or zero.



Standalone

In Context

Embed

Figure 13.14.11 The vector field $\vec{F} = \langle xy - 2z, y^2 - yz, 3x + z^2 \rangle$ in the region near Q

(b) Parametrize each of the surfaces S_{top}, S_{sides}, and S_{bottom}. Be sure to give bounds for each of your parametrization.

(c) Give inequalities in terms of cylindrical coordinates to describe Q.

(d) Set up and evaluate double integrals to calculate the flux of $\vec{F}$ through S_{top}, S_{sides}, and S_{bottom}.

(e) What is the net flux through the closed surface? Be sure to state if the net flux is in or out.

(f) Compute the divergence of $\vec{F}$ and use this to explain whether you think $\iiint_Q \operatorname{div}(\vec{F})\, dV$ will be positive, negative, or zero.

(g) Set up and compute the triple integral for $\iiint_Q \operatorname{div}(\vec{F})\, dV$.

(h) Verify that your answers for part e and part g are the same and thus that the Divergence Theorem holds for this example.

(i) Was the left-hand side or right-hand side of the equation in the Divergence Theorem more tedious to calculate in this example? Do you think this will be true for most other cases where the Divergence Theorem applies?

Activity 13.14.3.

(a) Find the flux of the vector field $\vec{F} = \langle 3x^2 + y^5, 5 + e^{z^3}, z\rangle$ through the surface of the solid cube Q in $\mathbb{R}^3$ with $-2 \leq x \leq 2$, $-2 \leq y \leq 2$, and $-2 \leq z \leq 2$.

(b) Find the flux of the vector field $\vec{G} = \langle x^3, y^3, z^3\rangle$ through surface consisting of the top half of sphere of radius 3 centered at the origin and the disc of radius 3 in the xy-plane (centered at the origin).